Planning, Prices and Increasing Returns [1,2]

G. M. HEAL
Christ's College, Cambridge

I. INTRODUCTION

This paper presents a decentralized[3] planning routine—i.e. an iterative procedure for solving the problem of finding an optimal short-run plan for a centrally-planned economy. Earlier publications in this area group neatly into two distinct categories: those where the centre quotes prices for goods and services, and those where it proposes quantitative allocations.[4] The routine to be outlined below falls into neither of these categories. It can be given different institutional interpretations, one involving a mixture of price and command planning and bearing an appealing resemblance to the mixture of materials balances and markets actually used in certain planned economies, and the other partially resembling the market socialism of Lange, Arrow and Hurwicz.

The interesting features of the procedure are three-fold: it will locate a local maximum of the objective function even in the presence of increasing returns to scale, it satisfies Malinvaud's feasibility and monotonicity criteria, and has some of the informational economy of price-guided procedures. It also has limitations, connected with the generality of the model to which it is applicable. This point will become clear as the argument is developed, and will be discussed at length in the Appendix.

The difficulties that can arise when a market-like procedure is operated in an environment containing increasing returns are well-known. The analysis that follows suggests that if agents in the economy are required only to raise rather than to maximize the magnitudes in which they are interested, then this gives a market-like procedure added stability, and allows it to converge to an optimum even in the presence of increasing returns. An intuitive discussion of this point is contained in section VI.

II. THE MODEL

The model within which the planning procedure will operate may be formalized as follows. The only inputs to the production process are resources: these are used exclusively as inputs to production, and are not themselves produced. They are indexed by $j \in M$, $M = \{1, ..., m\}$. There are n firms, indexed by $i \in N$, $N = \{1, ..., n\}$, and p distinct produced goods, indexed by $g \in P$, $P = \{1, ..., p\}$.

Our notation is as follows: X_{ij} is the amount of resource j allocated to firm i, Y_{ig} is the amount of good g produced by firm i, $R_j > 0$ is the total amount of resource j available to the economy, $\underline{X}_i$ is the vector of inputs to firm i, and $\underline{Y}_i$ is the vector of outputs of firm i.

1 *First version received March* 1970; *final version received Nov.* 1970 (*Eds*).
2 In writing this paper, I have been helped considerably by discussions with Christopher Bliss, Frank Hahn and Peter Hammond. Of course, none of these gentlemen should be held responsible for any errors that remain.
3 Decentralised is used here in the sense of " informationally decentralised ": this is the " classical " sense of Barone, Hayek and Lange. The alternative interpretation—decentralization of the authority to make decisions—is one that features rarely in the literature. This is perhaps unfortunate, as it is this interpretation that corresponds most nearly to the every-day usage of the term.
4 In the first category come [1], [7]: in the second, [4], [5] and [10]. Marglin's contribution [8] is an exception to this rule: it is in some respects similar to the present paper.

The production possibilities of firm i are represented by an implicit function:

$$T_i(\underline{Y}_i, \underline{X}_i) \leqq 0.$$

where $\underline{Y}_i$ and $\underline{X}_i$ are the vectors defined above.[1] It will be assumed that the set of efficient production programmes open to firm i can be represented by

$$T_i(\underline{Y}_i, \underline{X}_i) = 0.$$

It is also assumed that the T_i are once continuously differentiable.

In the ensuing argument we shall make frequent use of a slightly unconventional derivative: we shall use the symbol F^i_{gj} to stand for the rate at which firm i's output of good g changes, as the input of good j to firm i is varied, assuming that the quantities of the firm's various outputs are maintained in their existing proportions to each other. It is fairly easy to derive an expression for this derivative in terms of the conventional partials of T_i: in the derivation, we shall assume that both before and after a marginal change, the firm's production programme is efficient, i.e.

$$T_i(\underline{Y}_i, \underline{X}_i) = 0.$$

As subsequent discussion aims to show, such an assumption will not prove unduly restrictive.

Suppose that the amount of resource j allocated to firm i changes by ΔX_{ij}, and that there is as a result a proportionate increase of amount y in the output of each produced good. Then

$$T_i((1+y)\underline{Y}_i, X_{i1}, \ldots, X_{ij}+\Delta X_{ij}, \ldots, X_{im}) = \sum_{g \in P} yY_{ig}\partial T_i/\partial Y_{ig} + \Delta X_{ij}\partial T_i/\partial X_{ij}$$

to a first order approximation: this is zero if

$$y/\Delta X_{ij} = -(\partial T_i/\partial X_{ij})/(\sum_{g \in P} Y_{ig}\partial T_i/\partial Y_{ig}).$$

As $y = \Delta Y_{ig}/Y_{ig}$ where ΔY_{ig} is the relevant change in i's output of g,

$$F^i_{gj} = -(Y_{ig}\partial T_i/\partial X_{ij})/(\sum_{g \in P} Y_{ig}\partial T_i/\partial Y_{ig}). \qquad \ldots(1)$$

This equality is well-defined as long as the Y_{ig} are not zero for all g: in such a case, the Y_{ig} can be assigned arbitrary values, though the derivatives must still be evaluated at $\underline{Y}_i = 0$.

We need to make one further, fairly innocuous, assumption about firms' production possibilities—a " finite input, finite output " assumption.

Formally, we assume that if $\| \underline{X}'_i \| < A$, where A is finite, then any $\underline{Y}'_i$ satisfying

$$T_i(\underline{Y}'_i, \underline{X}'_i) = 0$$

also satisfies

$$\| \underline{Y}'_i \| < B, \qquad \ldots(A1)$$

for some finite B.

The symbol $\| \underline{X}'_i \|$ denotes the Euclidean norm of the vector $\underline{X}'_i$.

Let $Y_g = \sum_{i \in N} Y_{ig}$ etc. Then the objective of the planning procedure can be specified as follows:

Maximize $U(Y_1, \ldots, Y_p)$ subject to

$$T_i(\underline{Y}_i, \underline{X}_i) \leqq 0 \quad \text{for all } i \in N. \qquad \ldots(2A)$$

$$\sum_{i \in N} X_{ij} \leqq R_j \quad \text{for all } j \in M. \qquad \ldots(2B)$$

$$X_{ij}, Y_{ig} \geqq 0 \quad \text{for all } i, g, j. \qquad \ldots(2C)$$

[1] This notation allows us to deal conveniently with the case of joint production when substitution between outputs is possible. A production programme for firm i where Y_i is maximal under $\geqq$ for given X_i, or where X_i is minimal under $\geqq$ for given Y_i, will as usual be described as efficient for firm i.

U is a function of class C^1 from R^p to R^1, and has finite first partial derivatives.[1] ...($A2$)

We denote

$$\partial U / \partial Y_g \text{ by } U_g, \text{ etc.}$$

III. THE PLANNING PROCEDURE

Before describing the details of the planning procedure, it is necessary to introduce one additional concept—the value of a resource in a particular use. The value of resource j in firm i, V_{ij}, is given by

$$V_{ij} = \sum_{g \in P} U_g F^i_{gj} \qquad \text{...(3)}$$

and thus gives the rate at which U would change if X_{ij} were changed marginally, and firm i maintained its existing output proportions. It is, in a sense, a " shadow price " for the variable X_{ij}.

As mentioned in the introduction, there are a number of different institutional interpretations that can be given to the planning procedure under consideration. However, in all variants, the differential equations governing the reallocation of resources are the same: this provides the justification for speaking of different interpretations of one planning procedure, rather than about three distinct procedures.

III. (1) Price-and-Command Planning

In this version of the planning procedure, economic activity is controlled partly by input quotas set by the central planning board (CPB), and partly by the use of output prices, also set by the CPB.

There are two main elements in the planning procedure:

(1) The re-allocation of resources amongst firms. This is carried out by the C.P.B. in the light of the V_{ij}: it increases the allocation of a resource to a firm where its value is above average, and vice versa.

(2) The substitution of one output for another. This is carried out by firms: at each stage of the process, the CPB announces " prices " for each produced good—the price of good g is U_g, the derivative of the objective function w.r.t. the output of that good at the current output levels. Taking these prices, and its inputs of resources, as given, each firm then adjusts its output mix so as to increase the value of its output.

Details of the planning process are as follows. Starting from an arbitrary feasible plan satisfying ($2A$) and ($2B$) with equality,

(1) Firms inform the CPB of their outputs of the various produced goods.

(2) The centre computes the totals Y_g, and the prices, U_g, for $g \in P$, and informs firms of the latter.

(3) Each firm now calculates a value for every resource in its productive processes, and informs the CPB of these. (Alternatively, firms may inform the CPB of the quantities F^i_{gj} and leave the CPB to calculate the V_{ij}).

(4) The centre now changes the allocation of inputs amongst firms according to the following rules:

$$\dot{X}_{ij} = \begin{cases} V_{ij} - Av(K_j)V_{ij} \text{ for } i \in K_j \\ 0 \text{ otherwise.} \end{cases} \qquad \text{...(4)}$$

where a dot over a variable denotes its time derivative, and the notation $Av(K_j)V_{ij}$ denotes

[1] R^p is p-dimensional Euclidean space.

the average of the values of V_{ij} over the subscripts i contained in the set K_j. The set K_j is constructed iteratively as in [4], p. 350. It is defined by the following property

$$K_j = \{i: X_{ij} > 0 \text{ or } X_{ij} = 0 \text{ but } V_{ij} > Av(K_j)V_{ij}\}.$$

and contains only firms whose allocation of resource j is positive, or those whose allocation is zero but where the value is above the average over K_j. Hence application of equations (4) will never violate the non-negativity constraints.

(5) At the same time, each firm, remaining on the efficient surface given by the current input vector, substitutes between outputs so as to increase the total value of its output. That is, if $\dot{Y}^s_{ig}$ is the rate of change of i's output of g due to substitution between outputs, then the $\dot{Y}^s_{ig}$ are chosen so that the Y_{ig} vary continuously and

$$\sum_{g \in P} U_g \dot{Y}^s_{ig} \geqq 0, \quad \text{with equality if and only if the necessary conditions for a maximum of the value of output at prices } U_g \text{ are satisfied.} \qquad \ldots(5)$$

This completes one step of the process: we now return to item (1).

The necessary conditions referred to in item (5) can easily be derived. The relevant maximization problem is

$$\text{Maximize} \sum_{g \in P} U_g Y_{ig} \text{ subject to } T_i(\underline{Y}_i, \underline{X}_i) = 0,\ \underline{Y}_i \geqq 0, \text{ and } U_g \text{ and } \underline{X}_i \text{ given.}$$

which yields as necessary conditions

$$U_g - \mu_i \partial T_i / \partial Y_{ig} \leqq 0, \text{ with equality if } Y_{ig} > 0. \qquad \ldots(6)$$

(6) must hold for all $g \in P$, each $i \in N$.

Note that the total change in firm i's output of good g is the sum of any effect due to substitution between outputs and any effect due to changes in inputs, governed by (4). Hence the total is

$$\dot{Y}_{ig} = \sum_{j \in M} \dot{X}_{ij} F^i_{gj} + \dot{Y}^s_{ig}. \qquad \ldots(7)$$

III. (2) A " Market Socialist " Economy—(A)

In the process just described, the role of the centre can be given an interesting alternative interpretation, one more in line with " market socialism " of the type considered by Lange [6] and Arrow and Hurwicz [1]. Instead of supposing the centre to be allocating resources as in a command economy, we may imagine the centre to consist of a set of auctioneers, one for each resource. At each stage of the planning process, each auctioneer makes a prospective allocation of the resource at his disposal amongst firms. Starting from an arbitrary but feasible [i.e. satisfying (2B), in fact with equality] initial allocation of the resource, he invites bids for it and then changes the prospective allocation marginally in favour of higher bidding firms, and vice versa.

The quantity V_{ij} can be regarded as the price that firm i is prepared to bid for a marginal increment of resource j: for $V_{ij}\Delta X_{ij}$ is the contribution made to the value of firm i's output by an increment ΔX_{ij} of resource j, at current output prices and proportions. Thus according to this interpretation, firms bid for resources, and their bids, V_{ij}, are the maximum amounts that they would be prepared to pay for a marginal increment of the resource in question, given

(i) the objective of increasing, or at least not decreasing, the net value of the firm's production plan at current output prices.
(ii) the current output proportions.
(iii) the current prospective allocation of resources.

In response to these bids, the auctioneers alter the distribution of the resource at their disposal, away from low-bidding to high-bidding firms. At any given set of bid prices, each auctioneer changes his prospective allocations so as to increase his total revenues—i.e. he chooses the $\dot{X}_{ij}$ so that for his particular resource j,[1]

$$\sum_{i \in N} \dot{X}_{ij} V_{ij} \geqq 0, \qquad \ldots(8)$$

with equality if and only if the current allocation maximizes his revenue at the current bid prices: the condition for this is that for some scalar λ_j,

$$\left.\begin{array}{l} V_{ij} = \lambda_j \text{ for } i \text{ such that } X_{ij} > 0 \\ V_{ij} \leqq \lambda_j \text{ otherwise} \end{array}\right\}. \qquad \ldots(9)$$

Clearly there are many ways in which auctioneers could shift resources so as to satisfy (8): one such way would be to use equations (4) to determine the exact changes. For then by Lemma 1 of [4],

$$\sum_{i \in N} \dot{X}_{ij} V_{ij} = \sum_{i \in K_j} V_{ij}(V_{ij} - Av(K_j)V_{ij}) \geqq 0,$$

with equality if and only if conditions (9) hold. As the planning process continues, firms' bid prices alter, and auctioneers continually shift resources towards the currently highest bidders.

Firms' output mixes are determined as before: there is a central board which is informed of outputs, and sets output prices equal to the derivatives of the objective function. Firms then adjust their output mixes so as to raise the value of output at these prices.

On this interpretation, the planning procedure begins to look like something that could be called " market socialism ": firms facing given output prices adjust their output mixes, and bid for inputs, in such a way as to increase their profits. At the same time, auctioneers reallocate their wares amongst firms so as to raise their total returns at the bid prices. Of course, there is an important difference from a market procedure: neither firms nor auctioneers act so as to maximize the quantities in which they are interested, but act merely to increase them. There is, in a sense, friction present by comparison with a normal market procedure: however, it will be shown below that this friction is of crucial importance in dealing with non-convexities. If firms maximized profits at current prices, rather than merely acting so as to increase them, then the process would no longer converge to an optimum in the presence of increasing returns. A second difference from a market procedure is clearly to be found in the manner in which prices are determined: not by the forces of supply and demand, as those are always equated, but by reference to the marginal social valuations of goods. They might be described as " social demand prices ". Section IV of Arrow and Hurwicz [1], which to the best of my knowledge constitutes the only other published work on price-guided planning processes and increasing returns, also uses price-formation rules which do not correspond to those normally postulated for a market.

III. (3) A " Market Socialist " Economy—(B)

There is one further interpretation that may be given to the process under consideration: instead of supposing resources to be allocated by command, or auctioned towards the highest bidder, we can imagine that firms adjust their demands for inputs according to the marginal profitability of those inputs at certain prices currently quoted for them.

Firms again report the outputs that they produce at any input configuration to the centre, which quotes output prices equal as before to the derivatives of the objective function: each firm then substitutes between its outputs so as to raise the value of its output

[1] The auctioneer will of course respect a non-negativity constraint on his prospective allocations. He will also ensure that each proposed allocation satisfies the constraint (2B), and choose the $\dot{X}_{ij}$ so that the X_{ij} are continuous functions of time.

bundle at these prices. Firms also report to the centre the quantities V_{ij} corresponding to their current production programmes and the current output prices. The centre then calculates a price P_j for each resource given by $P_j = Av\,(K_j)V_{ij}$, where the set K_j is again constructed as in [4]. Firms then adjust their demands for resources according to the marginal profitabilities of those resources at current prices. Hence the rate of change of firm i's demand for resource j will be given by

$$\dot{X}_{ij} = V_{ij} - Av(K_j)V_{ij}. \qquad \ldots(10)$$

Of course, firms respect a non-negativity constraint on their inputs: hence (10) applies only to $i \in K_j$. Otherwise $\dot{X}_{ij} = 0$. We suppose the planning procedure to start from a feasible initial allocation of resources that satisfies the constraint (2B) with equality.

In this version of the planning procedure, there is by comparison with the earlier two some reduction in the amount of information to be transmitted by the centre. Firms still have to inform the centre of outputs, and of the quantities V_{ij} for all i and j: however, the centre no longer needs to specify an input allocation to each firm. Instead, it simply announces one price vector: it sends the same message to every firm, rather than sending n district messages to the n different firms.[1]

This completes our discussion of the planning procedure in its various manifestations: we now turn to a discussion of its properties. Section IV establishes the conditions that characterize a critical point of the planning problem, and section V proves and interprets certain convergence results.

IV. THE CHARACTERIZATION OF A CRITICAL POINT[2]

The planning problem we are concerned with can be formulated as follows:

Maximize $U(Y_1, \ldots, Y_p)$ subject to

$$T_i(\underline{Y}_i, \underline{X}_i) = 0 \quad \text{for all } i \in N.$$

$$\sum_{i \in N} X_{ij} \leqq R_j \quad \text{for all } j \in M$$

$$X_{ij}, Y_{ig} \geqq 0 \quad \text{for all } i, j \text{ and } g.$$

By formulating the problem in this way, we are assuming that a necessary condition for optimality is that firms should operate on their efficient surfaces. As long as " goods are goods " and have positive shadow prices, this will clearly be the case, and in fact it continues to be so even if some of a firm's outputs have negative values. In such situations the offending outputs can be reduced by substituting others for them, and/or reducing inputs. To operate within the efficient surface will always involve inefficiency in the sense of wasting some good with a positive shadow price, except in the trivial case when none of a firm's inputs or outputs have positive shadow prices. Forming the Lagrangean[3]

$$L = U(Y_1 \ldots Y_p) - \sum_{i \in N} \mu_i T_i(\underline{Y}_i, \underline{X}_i) + \sum_{j \in M} \lambda_j (R_j - \sum_{i \in N} X_{ij})$$

1 This type of point is discussed in some detail by Marglin [8]: in his paper, he makes an ingenious suggestion for reducing the information exchanges yet further. In the context of the present discussion, the impact of his proposal would be to relieve the firms of the need to inform the centre of the V_{ij}. His proposals are set out in detail on page 62: their relevance to the present discussion is clear.

2 By critical point is meant a local maximum, a local minimum, or a point of inflexion of the planning problem.

3 We shall assume the constraints $\sum_{i \in N} X_{ij} \leqq R_j$, for $j \in M$, to be satisfied with equality from now on. If the possibility of free disposal is available, this can certainly cause no embarrassment: otherwise, one needs to introduce explicit disposal activities.

we find that a point is a critical point if and only if

$$U_g - \mu_i \partial T_i / \partial Y_{ig} \leqq 0, \text{ with equality if } Y_{ig} > 0 \qquad \text{...(11)}$$

$$-\mu_i \partial T_i / \partial X_{ij} - \lambda_j \leqq 0, \text{ with equality if } X_{ij} > 0. \qquad \text{...(12)}$$

(11) of course holds for all i and g, and (12) for all i and j.

V. CONVERGENCE

Theorem 1. *If the production relations T_i and the objective function satisfy the assumptions of Section II, and if the initial allocation satisfies constraints (2A) to (2C) and is not a local minimum, then*

(a) *Every limit point of the re-allocation processes discussed in* III. (1) *to* III. (3) *is a critical point: such limits points exist and are not local minima.*

(b) *Along the paths produced by the processes of* III. (1) *to* III. (3), *the objective function increases monotonically.*

(c) *Every proposed allocation satisfies the constraints (2A) to (2C).*

Proof. Equation (5) generates feasible output vectors (i.e. output vectors satisfying (2*A*)), from feasible output vectors: that equation (4) generates feasible allocations (i.e. allocations satisfying (2*B*)) from an initially feasible allocation, was established in [4], equation (15). The construction of the set K_j (see [4], following equation (13)) ensures that if the non-negativity constraints were initially satisfied, then they will always be satisfied subsequently. Hence (*c*) of the Theorem follows. We next establish (*b*).

$$\dot{U} = \sum_{g \in P} U_g \sum_{i \in N} \dot{Y}_{ig}$$

and from (7) this gives, after rearrangement,

$$\dot{U} = \sum_{j \in M} \sum_{i \in K_j} \dot{X}_{ij} \sum_{g \in P} U_g F^i_{gj} + \sum_{g \in P} \sum_{i \in N} U_g \dot{Y}^s_{ig}$$
$$= \sum_{j \in M} \sum_{i \in K_j} \dot{X}_{ij} V_{ij} + \sum_{g \in P} \sum_{i \in N} U_g \dot{Y}^s_{ig}.$$

The second term on the R.H.S. is non-negative by the construction of the firms' output changes (equation (5)). In the case of process III. (2), the first term is also non-negative by construction (equation (8)): in the other cases its non-negativity follows from Lemma 1 of [4]. Indeed equations (4), (5), (6), (8), (9) and this Lemma imply that $\dot{U} \geqq 0$, with equality if and only if

$$\left.\begin{array}{ll} V_{ij} = V_{kj} = \lambda_j & \forall i, k \in K_j \\ V_{ij} \leqq \lambda_j & \forall i \notin K_j \end{array}\right\}, \qquad \text{...(13)}$$

$$U_g - \mu_i \partial T_i / \partial Y_{ig} \leqq 0, = \text{if } Y_{ig} > 0. \qquad \text{...(14)}$$

where (13) holds for all $j \in M$, and (14) for all $g \in P$, and each $i \in N$. (14) is clearly identical with equation (11) in section IV. (13) can be rewritten, using the definitions (3) of the V_{ij} and (1) of the F^i_{gj} as

$$-\Big(\sum_{g \in P} U_g Y_{ig} \partial T_i / \partial X_{ij}\Big) \Big/ \Big(\sum_{g \in P} Y_{ig} \partial T_i / \partial Y_{ig}\Big) \leqq \lambda_j, \qquad \text{...(15)}$$

with equality if $X_{ij} > 0$.

From the complementary slackness condition in (14), (15) becomes

$$-\mu_i \partial T_i / \partial X_{ij} \leqq \lambda_j, \text{ with equality if } X_{ij} > 0$$

which is of course (12), the second of the conditions of section IV. The following proposition has therefore been established: $\dot{U} \geqq 0$, with equality if and only if a critical point has been reached, and (*b*) of the Theorem follows.

It remains only to prove (*a*). Let the vector $\underline{x}$ denote the state of the economy—it is a vector of inputs to and outputs from all firms. The constraints (2*A*) to (2*C*), and (*A*1), imply that the set F of feasible $\underline{x}$ is compact. U therefore attains its maximum, which by (*A*2) is finite, on this set: call it U_m. Point (*c*) of the Theorem implies that every trajectory of the reallocation process lies in F if its initial point is in F. Denote by $T(\underline{x}_0)$ the trajectory traced out by the reallocation equations from some initial $\underline{x}_0 \in F$: i.e. $\underline{y} \in T(\underline{x}_0)$ if and only if there exists a $t \geqq 0$ such that $\underline{x}(t) = \underline{y}$. Define the positive limit set of the trajectory through $\underline{x}_0$, $T_\infty(\underline{x}_0)$, as

$$T_\infty(\underline{x}_0) = \bigcap_{\underline{y} \in T(\underline{x}_0)} \overline{T(\underline{y})}$$

where a bar over a set denotes its topological closure. Now the closure of the trajectory $T(\underline{x}_0)$ is a compact set: it follows that $T_\infty(\underline{x}_0)$ is not empty, and, by an argument analogous to that in [9], pp. 340-341, that the trajectory through $\underline{x}_0$ converges to $T_\infty(\underline{x}_0)$.[1] Define a function $V(\underline{x})$ from F to the reals:

$$V(\underline{x}) = U_m - U(\underline{x})$$

$V(\underline{x})$ is continuous, finite, bounded below by zero, and non-increasing: hence there exists an $\alpha \geqq 0$, depending on $\underline{x}_0$, such that along the trajectory $T(\underline{x}_0)$, $V(\underline{x}) \to \alpha$ as $t \to \infty$. It follows by continuity that $V(\underline{x}) = \alpha$ on the set $T_\infty(\underline{x}_0)$. Hence $\dot{V}(\underline{x}) = 0$ on this set, and so $T_\infty(\underline{x}_0)$ contains only critical points. It is thus established that the process converges to a set containing only critical points: such points cannot be local minima as they are approached along a path where $\dot{U} > 0$. Q.E.D.

In all plans proposed by the reallocation process, constraint (2*B*) will be satisfied with equality. As in [4], the introduction of disposal activities will deal with any problems that this might cause.

Theorem 1 established the existence of limit points to the solution paths of the reallocation equations: it did not establish that any such solution path has a unique limit point. However, it is possible to strengthen slightly the results of the Theorem. It is clear that the set of limit points of any solution path contains only singularities of the differential equations: this rules out the possibility of limit cycles. It is also clear that this set must be connected ([3], p. 306): hence if the points satisfying the necessary conditions for optimality are isolated, any solution path can have only one limit point. And the relevant points will of course be isolated, provided that there are at most countably many of them.

VI. DISCUSSION

It may at first sight seem surprising that a procedure which makes use of prices can locate a local optimum even in the presence of increasing returns in production. The problems that normally arise are well-known, and illustrated by figure 1. This refers to a one-firm economy using a single input to produce two outputs, A and B. The transformation curve between A and B is drawn continuous, broken lines show contours of the objective function, the optimum is at Q, and the line PP' has a slope equal to both the marginal rates of transformation and substitution at Q.

[1] It is implicit in this argument that there is a unique and continuous solution to equations (4). This is not immediately obvious, as there is a discontinuity in the right-hand side at times when the composition of any of the sets K_j changes. But for the time-intervals during which the compositions of the sets K_j are constant, the equations are well-behaved and yield unique and continuous solutions. The solution for all time is then obtained by piecing together a series of such solutions—as in Arrow, Hurwicz and Uzawa [2], chapters 6 and 7.

However, if the firm is asked to maximize its profits facing output prices whose ratio equals the slope of PP', it will produce at R rather than Q: Q cannot be reached by maximizing behaviour. We now examine how this problem is solved by the procedure outlined above. Suppose the firm to produce initially at S: then the price ratio quoted is given by the slope of the contour through S. Taking this price ratio as given, the firm alters its output vector so as to raise its profits: this involves moving along the transformation curve in the direction of the arrow at S. Similarly, at T the firm would face prices given by the slope of the contour at T, and to raise profits would move in the direction given by the arrow at T. It is easy to see that wherever the firm is on the transformation curve, it will be facing prices

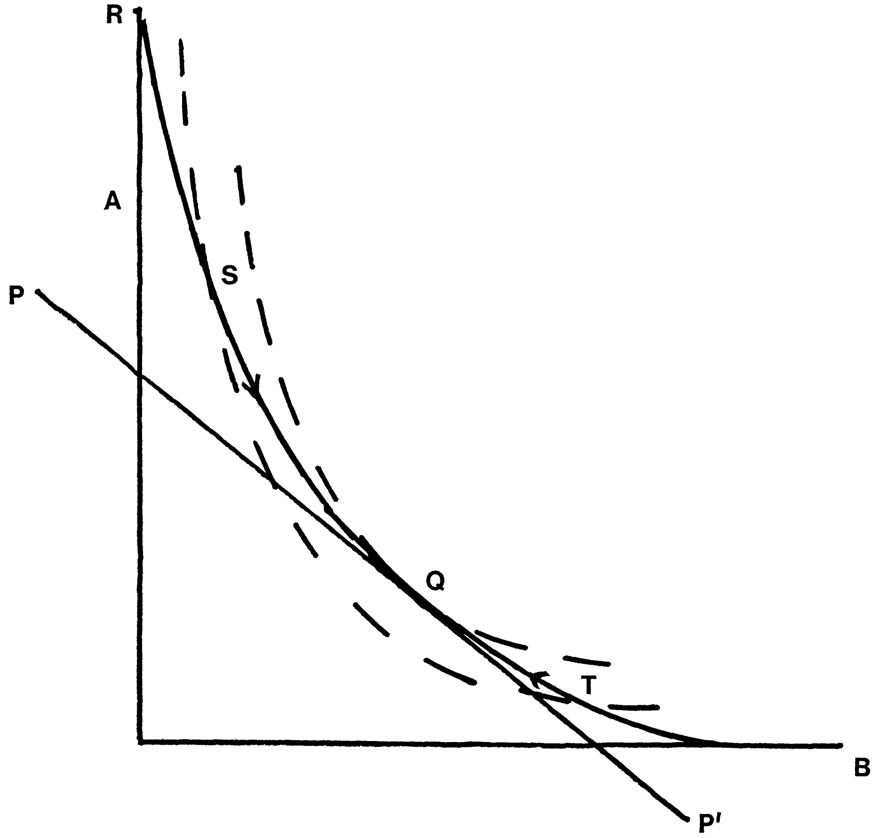

FIGURE 1

that induce it to move towards Q: gradual alterations to its output vector so as to raise profits at current prices will lead it to Q. However, if instead of making gradual adjustments it acted so as to maximize the profits at current prices, it would clearly oscillate between opposite ends of the transformation curve.

Hence the importance of " friction ", referred to earlier.[1]

One further point is worthy of discussion. The market-like procedure discussed in III. (3) has the unusual property for a market that the total quantity of inputs allocated remains constant: hence, given a feasible initial allocation, the procedure satisfies Malinvaud's feasibility criterion. But suppose, one might reasonably ask, that the initial allocation was infeasible, in that it involved non-zero excess demands. Could the procedure in

[1] A similar point arises in connection with the procedure for allocating resources. In III. (3), firms increase their demand proportionally to marginal profitabilities: if they were to announce demands that maximized profits, it is clear that these could under certain circumstances be infinite.

T

III. (3) be adapted to clear the markets in this case? The obvious way to tackle this problem would seem to be to define the price P_j as

$$P_j = Av(K'_j)V_{ij} + \alpha_j(\sum_{i \in N} X_{ij} - R_j)$$

where $\alpha_j > 0$ is a speed-of-adjustment parameter, and K'_j is given by

$$K'_j = \{i\colon X_{ij} > 0 \text{ or } X_{ij} = 0 \text{ but } V_{ij} > Av(K'_j)V_{ij} + \alpha_j Z_j\}.$$

Z_j denotes the excess demand for resource j: thus $Z_j = \sum_{i \in N} X_{ij} - R_j$. It now follows that if $|K'_j|$ is the number of elements in the set K'_j,

$$d(\tfrac{1}{2}Z_j^2)/dt = Z_j \sum_{i \in K'_j} \dot{X}_{ij} = -|K'_j|\,\alpha_j(Z_j)^2 \leqq 0,$$

with equality if and only if $Z_j = 0$. One can now establish convergence results similar to part (*a*) of Theorem 1—namely, that any limit point of the process is a critical point of the planning problem which is not a local minimum—though of course there are now no equivalents to parts (*b*) and (*c*) of the Theorem.

Although the above discussion was introduced in connection with the procedure outlined in III. (3), it should be clear that analogous modifications, with the same results, could be made to either of the other formulations of the planning procedure.

Before concluding the discussion of this paper, it is worth mentioning a problem of some substance that has not yet been raised—the problem of distinguishing local from global maxima.[1] This problem arises as soon as non-convexities are permitted, and it will be apparent that the method proposed in this paper does not contribute to its solution. It is essentially a gradient method, and climbs the nearest hill: this need not, of course, be the highest. In order to establish interesting global results, it would be necessary to combine some technique for discovering the general location of a global maximum with the routine discussed here, applying the latter from an initial point " near " the global maximum. An approximation to the overall maximum sufficiently close for this purpose, could perhaps be obtained by the CPB if it had some outline of the nature of production relations (and particularly of returns to scale) in the major sectors of the economy. Using this information, it could set up a relatively simple non-linear programming problem that captured some of the essential features of the true planning problem, and solve this on its own computers: the solution would give it some indication of where to start the decentralized routine.

VII. INCENTIVES

The foregoing analysis suggests a method of locating a socially-optimal production point in the presence of increasing returns, but does not unfortunately suggest a method of supporting such a point. If the firm in figure 1 were to take the output prices finally quoted as given and attempt to maximize profits at these, it might well be induced to move from the point Q. One is naturally led to ask whether there is some system of incentives that will lead firms to continue voluntarily to produce at the optimum: of course, they could just be ordered to produce at this point, but such an approach has many disadvantages. Some partial results on such a system of incentives follow: they suggest that although the optimum cannot be supported by given prices, it can be supported in some cases by prices that vary in an intuitively appealing way with output levels. (That is, it can be supported by demand schedules.) The analysis bears a considerable resemblance

[1] There is also the need to discuss a discrete rather than a continuous adjustment process. Fortunately it is not particularly difficult to establish that the earlier results apply in this case. If auctioneers and firms take steps of positive size chosen so as to give positive increments in revenues or profits respectively, then the discrete-step process also converges: it is always possible for the agents to make such steps, except at local maxima.

to that in section IV of Arrow and Hurwicz [1]—though the properties of the procedure in this paper are somewhat stronger.

The essentials of the argument can be appreciated in a one-firm model. Let the economy consist of a single firm: when using all the resources available (these are not consumable) it can produce an output vector $\underline{Y} = (Y_1, ..., Y_p)$ according to the equation $F(\underline{Y}) = 0$. The social welfare function is $U(\underline{Y})$. Suppose now that the firm is told that the use of resources is free, that it will be paid an amount U_g for each unit of good g it produces, and that the price vector $(U_1, ..., U_p)$ will vary as the output vector varies. The firm is then instructed to maximize its profits (which in this case equal its revenue): this puts it in the position of a monopolist, and it has to solve the problem

$$\max \sum_{g \in P} Y_g U_g \text{ subject to } F(\underline{Y}) = 0, \qquad \ldots(16)$$

where the vector $(U_1, ..., U_g)$ depends on $(Y_1, ..., Y_g)$ in a way that may or may not be known to the firm. Assume the social welfare function to have the following property:

$$\sum_{g \in P} Y_g U_g = \phi[U(\underline{Y})] \text{ for some } \phi \text{ with } \phi' > 0 \qquad \ldots(17)$$

then the solution to the problem (16) is identical to the solution to the planning problem

$$\max U(\underline{Y}) \text{ subject to } F(\underline{Y}) = 0, \qquad \ldots(18)$$

and the social optimum is the monopolist's profit-maximum. Condition (17) is by no means completely unacceptable: the class of positively homogeneous functions is a sub-class of those satisfying it. Three interesting implications result from this identity of the solutions to (16) and (18):

(i) During the application of any of the planning processes described above to this simple economy, the monopolist's profits will rise monotonically as the process continues.

(ii) Any departure from the socially optimal production plan will lower the firm's profits.

(iii) No false reporting of outputs during the procedure leading to the optimum could increase the total profit that accrues to the firm at the equilibrium.

These conclusions have to be modified for the many-firm model set out in section II of the paper. In that case one has corresponding to point (i) the fact that the total revenue from the sale of the outputs of all firms is rising during the planning process; in points (ii) and (iii) the " total revenue of all firms " must likewise be substituted for " firm's profits ".

It is also possible to make the following statement about this more general case. Consider an arrangement whereby once the optimum had been attained, firms were permitted to trade resources with each other and the auctioneers. Then the sum of the profits earned by all firms and auctioneers would be maximized, over all possible allocations, at the social optimum. The proof of this assertion is trivial: if P_j is the price of resource j, and π the sum of profits,

$$\begin{aligned} \pi &= \sum_{i \in N} \sum_{g \in P} Y_{ig} U_g - \sum_{i \in N} \sum_{j \in M} X_{ij} P_j + \sum_{j \in M} \sum_{i \in N} X_{ij} P_j \\ &= \phi[U(Y_1, ..., Y_p)] \text{ with } \phi' > 0 \text{ if (17) is satisfied.} \end{aligned}$$

It is therefore true that if in such a situation any agent departs from his socially optimal action the resulting gain to him must be less than the losses to others: the losers could therefore bribe the gainers not to make such a departure.

In summary, it is clear that although the social optimum located by the planning processes discussed cannot be supported by prices in the normal way, there is implicit in the rules of the economy a structure of incentives that, if the objective function satisfies (17), makes a departure from the social optimum against the interests of all agents taken together. By the same reasoning, false reporting during the planning process done with the intention of diverting the process to a point other than the social optimum, though it may be in the interests of a subset of agents, cannot be in the interests of all. Those who lose from such a diversion to a non-optimal point could profitably bribe those who gain from it not to cause it.

APPENDIX

A More General Case

The discussion above was presented entirely in the context of an economy with no intermediate goods: it can be extended to a more general model, though of course at the cost of complicating the discussion. This appendix outlines how the procedure of section III. (2) can be modified to cope with a more general—though not completely general— case. At first, however, it is worth noting that the model presented above was perhaps not as restrictive as it might appear, and must be interpreted carefully. The term " resource ", for example, was used in an unconventional way: namely, to denote any commodity whose supply was fixed within the planning period under consideration. It therefore refers not only to resources in the normal sense of the word, but also to any form of produced good whose production period exceeds the planning period. Similarly, the assumption that there are no intermediate goods implies only that none of the outputs of firms are used as inputs to production within the planning period: they may of course be stored and used as inputs in some subsequent period. It also emerges from this that the statement that a good is allocated to final demand must be interpreted generously: final demand may include storage for use as an industrial input in some subsequent period. For a planning period of the order of a year, it is clear that these qualifications reduce somewhat the severity of the initial model.

In the extension of procedure III. (2) to be considered here, there will be an auctioneer for each produced good, in addition to one per resource. The auctioneer for a particular produced good will buy the entire output of that good, and then auction it amongst the firms using it. Firms would submit bids for the good in the usual way: the price that an auctioneer paid for a good would depend upon the bids submitted to him for it. The present case thus differs from the earlier one in that the auctioneer is now both buying and selling, and is selling a varying total amount (because the outputs of produced goods change as the inputs to the firms producing them change.)

To discuss the planning routine precisely, it is necessary to specify the structure of the economy to which it is applied: it will be assumed that the planning period is chosen sufficiently short that the structure of inter-firm flows is " linear " or " Austrian ". In other words, firms can, within the planning period, be unambiguously identified as being at earlier and later stages of the production process: formally, the set of N firms can be divided into subsets N_1, N_2, ..., N_T such that firms in subset N_i (the " ith stage of production ") supply outputs only to firms in subsets N_{i+j}, j a natural number, or to final demand, and receive them only from those in N_{i-j}. The realism of such an assumption depends upon the time-period involved. Thus a firm in stage 2 might produce steel, and a firm in stage 3, vehicles: some of the vehicles produced in a given period might well be used as an input to the steel industry eventually, but with a limited slice of time—of the order of a year—it is unlikely that the vehicles produced in that period will be used as inputs to the steel industry in the same period. They might of course be so used subsequently: and in the period under consideration, vehicles produced earlier and stored might be used as inputs to the steel industry. The availability of such vehicles would be constant within the given planning period, and they would therefore count as resources.

In the setting of such a model, the planning procedure would work as follows. Starting from an arbitrary feasible plan, the CPB would announce prices for the goods allocated to final demand, equal as before to the derivatives of the objective function. Given these, firms supplying only final demand—i.e. firms at stage T—would have the prices of all their outputs determined, and could as before announce bid prices for inputs. Auctioneers dealing with goods produced by firms at stage $(T-1)$ would then announce the prices that they were prepared to pay for these: these prices could most conveniently be regarded as an average of the bid prices. In this manner, buying and selling prices could be established for all goods.

The subsequent reallocation of goods is most easily described as a series of discrete operations. Initially, the auctioneers controlling resources would change their prospective allocations in the usual way, choosing the changes ΔX_{ij} in the X_{ij} so that $\sum_{i \in N} \Delta X_{ij} V_{ij} \geqq 0$, with equality if and only if the initial allocation maximizes the sum $\sum_{i \in N} X_{ij} V_{ij}$. A firm in the first stage of production would then revise its output plan in two ways—firstly, so as to allow for the changed availability of inputs, and secondly, it would substitute between outputs so as to raise the values of its output bundle at current prices. The total change in its output bundle will thus be the composition of two changes, one along a ray in output space, and the other along the feasible surface in output space. The next step would be taken by the auctioneers who control the inputs used by firms at the second stage of production. They buy these goods, and reallocate them amongst firms in stage 2. They are of course reallocating an amount that differs from that associated with the initial plan, because the outputs of firms at stage 1 have been altered: let ΔY_g be this change in the total output of good g. Then there are two components to the change that the auctioneer makes in the allocation to a firm i. Call the first ΔY_g^i: then these terms are chosen so that

$$\sum_{i \in N} \Delta Y_g^i V_{ig} \geqq 0,$$

with equality if and only if the initial allocation maximizes the total revenue at the current bid prices. The second component is simply $1/\tau \Delta Y_g$, where τ is the number of firms whose allocations are being altered.[1] A firm's allocation therefore changes at a rate

$$(\Delta Y_g^i + 1/\tau \Delta Y_g):$$

the auctioneer changes the allocations so as to receive more revenue for a given total amount of the good, but has also to adjust for the fact that his total is changing.[2] This process continues along the stages of production until all firms have adjusted their inputs and outputs, at which point prices are revised and it is started again. If the steps made during the process are sufficiently small, it is clear that it represents a generalization of the continuous process of section III. (2), and one can therefore establish analogous convergence properties.

REFERENCES

[1] Arrow, K. J. and Hurwicz, L. " Decentralisation and Computation in Resource Allocation " in Pfouts, R. (Ed.) *Essays in Economics and Econometrics* (University of North Carolina Press, Chapel Hill).

1 The auctioneer will, of course, respect a non-negativity constraint.

2 This is the simplest story that can be told: we could instead require the auctioneer to change the allocations so as to maximize the rate of increase of his revenue subject to the constraint of keeping supply and demand equal—i.e. to choose the ΔY_g^i to maximize

$$\sum_{i \in N} \Delta Y_g^i V_{ig} \text{ subject to } \sum_{i \in N} \Delta Y_g^i = \Delta Y_g.$$

[2] Arrow, K. J., Hurwicz, L. and Uzawa, H. *Studies in Linear and Non-Linear Programming* (Stanford University Press, Stanford, California, 1958).

[3] Bhatia, N. P. " Semi-Dynamical Systems " in part III of Kuhn, H. W. and Szegö, G. P. (Eds.). *Mathematical Systems Theory and Economics.* Lecture Notes in Operations Research and Mathematical Economics (Springer Verlag, 1969).

[4] Heal, G. M. " Planning without Prices ", *Review of Economic Studies* (July 1969).

[5] Kornai, J. and Liptak, T. " Two-level Planning ", *Econometrica*, **33** (1963).

[6] Lange, O. " On the Economic Theory of Socialism ", *Review of Economic Studies*, **4**, 1936-37.

[7] Malinvaud, E. " Decentralised Procedures for Planning " in Malinvaud, E. and Bacharach, M. O. L. (Eds). *Activity Analysis in the Theory of Growth and Planning* (I.E.A./Macmillan, 1967).

[8] Marglin, S. A. " Information in Price and Command Systems of Planning " in Margolis, J. and Guitton, H. (Eds). *Public Economics* (I.E.A./Macmillan, 1969).

[9] Nemytskii, V. V. and Stepanov, V. V. *Qualitative Theory of Differential Equations*, Princeton Mathematical Series (Princeton University Press, Princeton, New Jersey, 1960).

[10] Weitzman, M. " Iterative Multi-level Planning with Production Targets ", *Econometrica* (January 1970).