Optimal Growth with Intertemporally Dependent Preferences

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Optimal Growth with Intertemporally Dependent Preferences [1, 2]

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1. INTRODUCTION

In the literature on optimal economic growth, considerable effort has been devoted to the analysis of models using a preference function which depends additively on consumption at the various dates within the planning period. The typical formulation is that used by Cass [2]:

$$J[c(\cdot)] \equiv \int_0^\infty e^{-\delta t} u[c(t)]dt, \qquad \ldots(1)$$

where $\delta \geqq 0$ is an exponential discount factor, $c(t)$ is the current consumption level at date t. Koopmans [10] has shown that this formulation of the preference function is implied by certain assumptions of existence, continuity, sensitivity, stationarity, boundedness and independence. Hicks [8, p. 261] has identified independence as the key assumption, arguing that it is counter-intuitive. Instead, he claims that there is normally a strong complementarity between consumption at successive moments. This view is widely held, but seldom practiced " because we do not know how to specify [complementarities] in an analytical manner which would be deemed *adequate* to the problem ". [3, pp. 340, 341]. The object of this paper is to investigate the effect of using a model of the sources of consumer satisfaction that is possibly more realistic than the usual formulation of equation (1). The essential point is that it introduces into the utility function a new variable, z, which may be interpreted either as the customary level of consumption, or as the expected level of consumption. Instantaneous satisfaction then depends both on instantaneous consumption and on the customary or expected consumption level. The justification for including such a variable is obvious: it is that the amount of satisfaction that a man derives from consuming a given bundle of goods depends not only on that bundle, but also on his past consumption and on his general social environment.[3]

This approach has considerable intuitive plausibility: but if further justification is needed, then it is easily found. For example, it is not uncommon for sociologists concerned with political changes during economic development to remark that a period of historically high consumption levels followed by a drop in consumption is more likely to cause social

1 *First version received June* 1972; *final version received August* 1972 (*Eds.*).

2 The genesis of this paper is somewhat unusual, since the authors did not learn of each other's work until both had submitted finished drafts for publication [7, 17]. We have chosen to combine our papers in order to eliminate duplication and present our subsequent joint results. We gratefully acknowledge financial support from the Ford Foundation and the National Science Foundation and helpful discussions with Kenneth Arrow, Tony Atkinson, Christopher Bliss, Partha Dasgupta, Clifford Hildreth, Ettore Infante, Tjalling Koopmans, Bill Nordhaus, Joseph Stiglitz and each other.

3 Von Weizsäcker [23] has recently explored a different aspect of this phenomenon.

discontent than is a period of uniformly low consumption levels: in the former case, the period of high consumption builds up high customary or expected consumption levels, and the decline, though it may be to levels that are historically high, produces a sharp fall in satisfaction. Davies, in [6, p. 6], stresses that this is a factor contributing to the incidence of revolutions.[1]

The general proposition that a man's attitude towards his present economic circumstances is conditioned, *inter alia*, by his past experience working through his expectations, is one that is also endorsed at length by Katona in his analysis of psychological aspects of economic behaviour [9, Chapter 4 on " Past Experience and Expectations "].

2. THE BASIC MODEL

To formalize the notions discussed above, the variable $z(t)$ is defined by

$$z(t) = \rho e^{-\rho t} \int_{-\infty}^{t} e^{\rho\tau} c(\tau) d\tau, \qquad ...(2)$$

where $\rho > 0$ and $c(\tau)$ is the average level of per capita consumption in the community at time τ. $z(t)$ is thus a weighted average of past consumption levels, with the weights declining exponentially into the past. The larger is ρ, the less weight is given to past consumption in determining $z(t)$, and vice versa. It is the variable $z(t)$ that is regarded as the customary or expected consumption level at time t. In setting up a formal model for the study of individual behaviour, it would clearly be desirable to make $z(t)$ depend not only on the individual's past experience, but also on the consumption habits of those with whom he might compare himself. However, in dealing with aggregate figures in a national planning problem, this consideration carries less force: perhaps one ought to make $z(t)$ depend on the consumption standards current in other countries with which nationals of the country concerned might have contact, but for the sake of simplifying an already complex problem this embellishment is omitted.[2]

Our criterion assumes the form

$$J(c(\cdot)) \equiv \int_{0}^{\infty} e^{-\delta t} u[c(t), z(t)] dt. \qquad ...(3)$$

This form of criterion functional is specific enough to obtain some definite results, while permitting the marginal utility of consumption at a particular date to vary with past and future consumption.

We shall make a few preliminary assumptions on the momentary utility function $u[c(t), z(t)]$.

(P. 1) $u_c(c, z) > 0$. An increase in current consumption with no change in past consumption will increase utility.

(P. 2) $u_z(c, z) \leqq 0$. An increase in past consumption with no change in current consumption will not increase utility and may cause it to fall.

[1] To be sure, there are many outbreaks of social discontent to which his explanation cannot be applied (for example, those documented by Cohn [5]): but the evidence that he produces suggests that expectations produced by past experience may be an important factor in determining social satisfaction. There is a distinct and equally influential analysis of the causes of outbreaks of social discontent, first propounded by de Tocqueville and echoed by Cohn [5]. It is most aptly summarized by its progenitor on [19, p. 214]. Davies' explanation suggests that expectations are formed from past experience by some simple extrapolation, whereas de Tocqueville's suggests a more complex relationship in which the elasticity of expectations with respect to recent experience may at certain crucial stages of development be very great indeed. In the model that follows, there is no attempt to give expression to this more complex set of possibilities.

[2] In fact, it would not be difficult to make at least some concession towards including it. If consumption standards in other countries were growing steadily at say four per cent per annum, then an upward trend at this rate could be included in z.

(P. 3) $u_c(c, c)+u_z(c, c)>0$. An increase in a uniformly maintained consumption level will increase utility.

(P. 4) $u_{cc}(c, z)<0$, $u_{cc}(c, z)u_{zz}(c, z)-[u_{cz}(c, z)]^2 \geqq 0$. Momentary utility is concave in c and z, strictly concave in c.

(P. 5) $\lim_{c\to 0} u_c(c, z) = \infty$ uniformly in z; $\lim_{c\to 0} [u_c(c, c)+u_z(c, c)] = \infty$

In Section 4 below, we shall relax the assumption (P. 3) of non-satiation. We may note that our criterion reduces to the independent case if $u_z(c, z) = 0$.

The basic model of the economy which we shall adopt is the familiar neoclassical model used by Cass [2]. We assume one good, two factors, constant returns to scale, exogenously and exponentially growing labour force, stationary technology, exponential depreciation, and Inada conditions. In short, we have

$$\dot{k} = f(k)-\lambda k-c, \qquad \text{...(4)}$$

$$0 \leqq c \leqq f(k), \qquad \text{...(5)}$$

where c is consumption per worker, $k>0$ is the capital-labour ratio, $\lambda>0$ is the sum of the growth rate of the labour force and the depreciation rate of capital, and f is the production function. The production function satisfies

(T. 1) $f(k)>0, f'(k)>0, f''(k)<0$ for all $k>0$, and $\lim_{k\to 0} f(k) = 0$, $\lim_{k\to\infty} f(k) = \infty$,

(T. 2) $\lim_{k\to 0} f'(k) = \infty$, $\lim_{k\to\infty} f'(k) = 0$.

Differentiating equation (2) with respect to time, we obtain a differential equation for z,

$$\dot{z} = \rho(c-z). \qquad \text{...(6)}$$

At the planning date, $t = 0$, we are faced with historically given endowments of capital $k_0>0$ and past consumption $z_0>0$. Thus we have the following problem of optimal control.

We wish to choose a consumption path $c(t)$, $t \geqq 0$ that will maximize

$$J[c(\cdot)] = \int_0^\infty e^{-\delta t} u[c(t), z(t)]dt \qquad \text{...(3)}$$

subject to

$$\dot{k} = f(k)-\lambda k-c \qquad \text{...(4)}$$

$$0 \leqq c \leqq f(k) \qquad \text{...(5)}$$

$$\dot{z} = \rho(c-z) \qquad \text{...(6)}$$

$$z(0) = z_0>0, \ k(0) = k_0>0. \qquad \text{...(7)}$$

3. COMPLEMENTARITY OVER TIME

To discuss marginal utilities and marginal rates of substitution between consumptions at various dates, we must use the concept of the derivative of a functional. This concept was worked out by Volterra [20, p. 23], and may be explained as follows.

We wish to measure the increment in $J[c(\cdot)]$ resulting from a small increment in consumption near date t_1. Let $c'(\cdot)$ be an infinite consumption stream satisfying

$$\left.\begin{aligned} &c'(t) = c(t) \text{ for } t \leqq t_1 - \frac{\beta}{2},\ t \geqq t_1 + \frac{\beta}{2}. \\ &|c'(t)-c(t)|<\alpha \text{ for } t_1 - \frac{\beta}{2} < t < t_1 + \frac{\beta}{2}. \end{aligned}\right\} \qquad \text{...(8)}$$

Either $c'(t) \geqq c(t)$ or $c'(t) \leqq c(t)$ for all t. Let

$$\varepsilon = \int_{-\infty}^{\infty} [c'(t)-c(t)]dt = \int_{t_1-\beta/2}^{t_2+\beta/2} [c'(t)-c(t)]dt. \quad ...(9)$$

Then $|\varepsilon| < \alpha\beta$. (See Figure 1.) The Volterra derivative of J is

$$J'[c(\cdot);\ t_1] = \lim_{\substack{\alpha\to 0\\ \beta\to 0}} \frac{J[c'(\cdot)]-J[c(\cdot)]}{\varepsilon}. \quad ...(10)$$

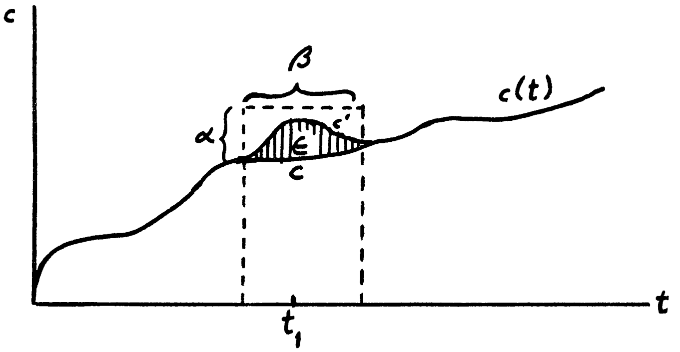

FIGURE 1

Volterra derivative.

The Volterra derivative is itself a functional which may be differentiated in the same way to obtain second and higher Volterra derivatives, $J''[c(\cdot);\ t_1, t_2]$. When welfare is a functional of a continuous stream of consumption, the marginal utility of consumption at date t_1 is the Volterra derivative $J'[c(\cdot);\ t_1]$. The marginal rate of substitution between consumption at dates t_1 and t_2 is the ratio of marginal utilities,

$$R[c(\cdot), t_1, t_2] \equiv \frac{J'[c(\cdot);\ t_1]}{J'[c(\cdot);\ t_2]}. \quad ...(11)$$

If there is a small increment to consumption at date t_3, the effect on the marginal rate at substitution between t_1 and t_2 is another Volterra derivative,

$$R'[c(\cdot), t_1, t_2;\ t_3] = (J'[c(\cdot);\ t_2]J''[c(\cdot);\ t_1, t_3]-J'[c(\cdot);\ t_1]J''[c(\cdot);\ t_2, t_3])/(J'[c(\cdot);\ t_2])^2 \quad ...(12)$$

If $R'[c(\cdot), t_1, t_2;\ t_3]>0$, a small increment at t_3 shifts preferences from t_2 to t_1. In this case we would have complementarity between t_3 and t_1. If $R'<0$, the increment at t_3 shifts preferences from t_1 to t_2, giving us a complementarity between t_3 and t_2. If $R' = 0$, an increment at t_3 does not affect preferences between t_1 and t_2. This would be the case for all $c(\cdot)$, t_1, t_2, t_3 if the preference functional is intertemporally independent. It should be noted that complementarity, as defined here, is different from complementarity in the Slutsky sense.

Taking derivatives of the functional defined by equations (3) and (2), we obtain for $t_2>t_1>0$

$$J'[c(\cdot);\ t_1] = e^{-\delta t_1}u_c[c(t_1), z(t_1)]+\rho e^{\rho t_1}\int_{t_1}^{\infty} e^{-(\rho+\delta)t}u_z[c(t), z(t)]dt. \quad ...(13)$$

$$J''[c(\cdot);\ t_1, t_2] = \rho e^{\rho t_1-(\rho+\delta)t_2}u_{cz}[c(t_2), z(t_2)] + \rho^2 e^{\rho(t_1+t_2)}\int_{t_2}^{\infty} e^{-(2\rho+\delta)t}u_{zz}[c(t), z(t)]dt. \quad ...(14)$$

Since the expressions obtained by substituting (13) and (14) into (12) are too complicated to be of much help, let us examine in particular the values obtained along a constant consumption path $c(t) = z(t) = z(0)$ for all t. In that case u_c, u_z, u_{cz}, u_{zz} become constants. Now we have for $t_2 > t_1 > 0$

$$J'[c;\ t_1] = e^{-\delta t_1}\left[u_c + \frac{\rho}{\rho+\delta}u_z\right]. \qquad \text{...(15)}$$

$$J''[c;\ t_1, t_2] = \rho e^{\rho t_1 - (\rho+\delta)t_2}\left[u_{cz} + \frac{\rho}{2\rho+\delta}u_{zz}\right]. \qquad \text{...(16)}$$

Under assumption (P. 3) we have $J' > 0$. Substituting (15) and (16) into (12), we obtain

$$R'[c, t_1, t_2;\ t_3] = \frac{\rho\left(u_{cz} + \dfrac{\rho}{2\rho+\delta}u_{zz}\right)}{u_c + \dfrac{\rho}{\rho+\delta}u_z} e^{\delta(t_2-t_1)}[\alpha(t_3-t_1) - \alpha(t_3-t_2)] \qquad \text{...(17)}$$

where $0 < t_1 < t_2$ and

$$\alpha(t) = e^{-(\rho+\delta)t} \quad \text{for } t > 0,$$

$$\alpha(t) = e^{\rho t} \quad \text{for } t < 0.$$

We see that R' and $\rho(u_{cz} + u_{zz}\rho/2\rho+\delta)$ have the same sign for $t_3 < ((\rho+\delta)t_1 + \rho t_2)/2\rho+\delta$ and opposite signs for $t_3 > ((\rho+\delta)t_1 + \rho t_2)/2\rho+\delta$. Figure 2 illustrates the pattern of intertemporal complementarity when $u_{cz} + u_{zz}\rho/2\rho+\delta > 0$. A positive value indicates complementarity between t_3 and t_1; a negative value indicates complementarity between t_3 and t_2. In this case we see that the proposed preference functional exhibits complementarity between adjacent dates. In the opposite case where $u_{cz} + u_{zz}\rho/2\rho+\delta < 0$, the signs would be reversed, giving us complementarity between distant dates instead.

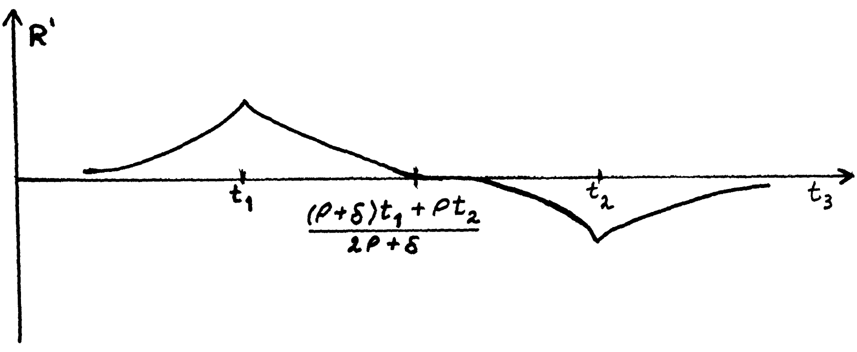

FIGURE 2

Complementarity pattern when $u_{cz} + \frac{\rho}{2\rho+\delta}u_{zz} > 0$.

In terms of the Wan-Brzeski example [21, p. 521] a person with distant complementarity who expects to receive a heavy supper would tend to eat a substantial breakfast and a light lunch. A person with adjacent complementarity would tend to eat a light breakfast and a substantial lunch in the same circumstances.

4. SATIATION OF UTILITY

When preferences are intertemporally independent, it seems reasonable to assume that more consumption is always preferred to less. In the present model, satiation is more plausible. If we compare steady states, then the customary level of consumption z must rise with the current level c. If the dissatisfaction engendered by higher expectations exceeds the enjoyment of higher realized consumption, the result will be a reduction in total utility.[1]

Figures 3 and 4 show the contours of the function $u(c, z)$ in the c-z plane. By assumption (P. 4) these contours bound convex level sets. Since $\dot{z} = 0$ if $c = z$, this line is the locus of steady states. We consider two cases.

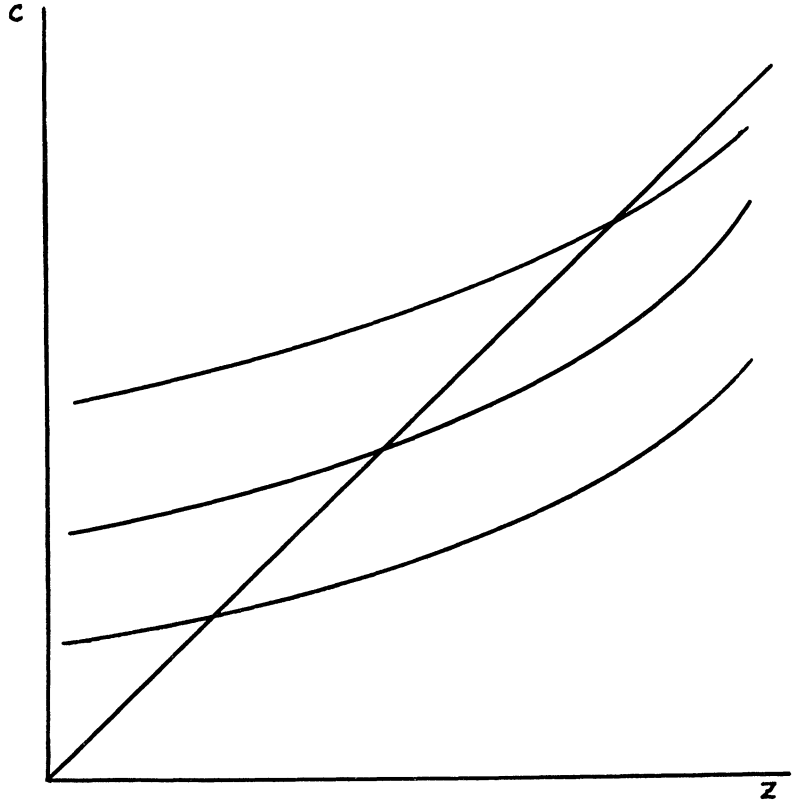

FIGURE 3

Indifference map under assumption (P. 3).

(i) Under assumption (P. 3) we have the configuration shown in Figure 3. No satiation is possible in this case.

(ii) If we relax assumption (P. 3) we may have the configuration shown in Figure 4. In this case we have satiation at $z = c = c_0$.

A satiated optimal stationary solution will exist if for some perpetually feasible c

$$J'(c;\ t_1) = e^{-\delta t_1}\left[u_c(c,\ c) + \frac{\rho}{\rho+\delta}\,u_z(c,\ c)\right] = 0. \qquad \text{...(18)}$$

The economic meaning of equation (18) is clear. At a satiated optimal stationary solution, the costs and benefits of a marginal temporary increase in consumption just cancel out. Hence, the stationary solutions defined by (18) are ones at which the economy will be indifferent between accepting and rejecting a marginal increment of consumption. Such

[1] It has also been suggested that one could interpret z as the stock of pollution resulting from past consumption—a very different interpretation from that in the text, but nevertheless one quite in keeping with the structure of the model. Equation (6) is then a radioactive decay equation for the pollutant. With this interpretation it becomes much easier to accept the possibility of satiation.

solutions could in principle occur in conventional optimal growth models, but they would require $u'(c) = 0$ for finite c: such a possibility is usually ruled out by assumption. In the present model it is more plausible to permit the total welfare impact of an increment of consumption to be zero, because there are costs to offset against the obvious benefits of an increase in consumption.

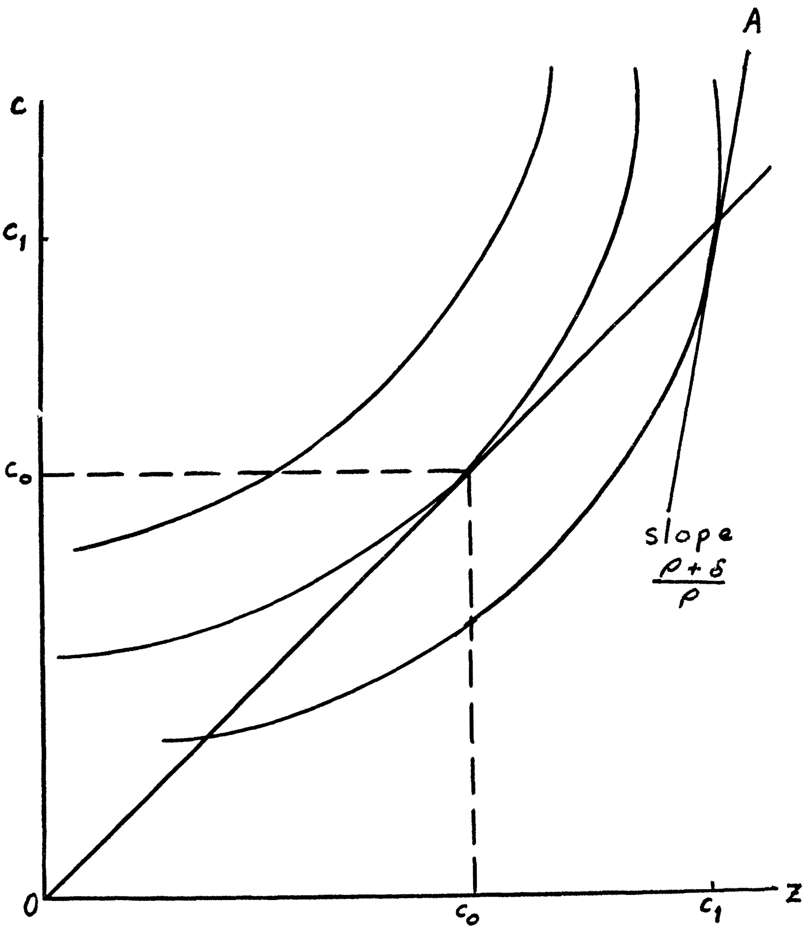

FIGURE 4

Indifference map when assumption (P. 3) is relaxed.

In Figure 4 a satiated optimal stationary solution will occur where the $c = z$ line is cut by an indifference curve with slope $(\rho+\delta)/\rho > 1$. This can only occur for $c > c_0$. Under distant complementarity, such a point is unique, but there is no such assurance under adjacent complementarity. To see this, let us define

$$q(c) \equiv u_c(c, c) + \frac{\rho}{\rho+\delta} u_z(c, c). \qquad \text{...(19)}$$

A satiated optimum occurs when $q(c) = 0$. But

$$q'(c) = \frac{(\rho+\delta)u_{cc} + (2\rho+\delta)u_{cz} + \rho u_{zz}}{\rho+\delta}. \qquad \text{...(20)}$$

Under distant complementarity, $q'(c) < 0$, but under adjacent complementarity, the terms in (20) may be offsetting. The concavity assumption (P. 4) implies

$$0 \geqq \left(\rho + \frac{\delta}{2}\right)^2 u_{cc} + 2\rho\left(\rho + \frac{\delta}{2}\right) u_{cz} + \rho^2 u_{zz}$$

$$= \frac{\delta^2}{4} u_{cc} + \rho(\rho+\delta)q'(c). \qquad ...(21)$$

Thus we have

$$q'(c) \leqq -\frac{\delta^2 u_{cc}}{4\rho(\rho+\delta)} > 0, \qquad ...(22)$$

which may allow multiple satiated optimal solutions as shown in Figure 5. By assumption (P. 5) we can assume that $q(c) > 0$ for small c, so the first satiated optimal solution must have $q(c_1) = 0$, $q'(c_1) \leqq 0$.

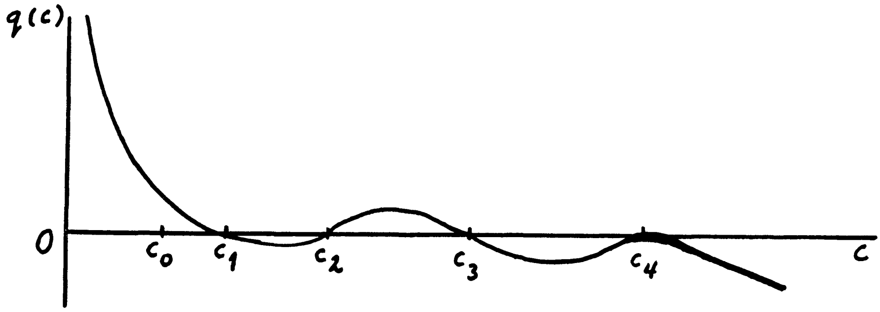

FIGURE 5

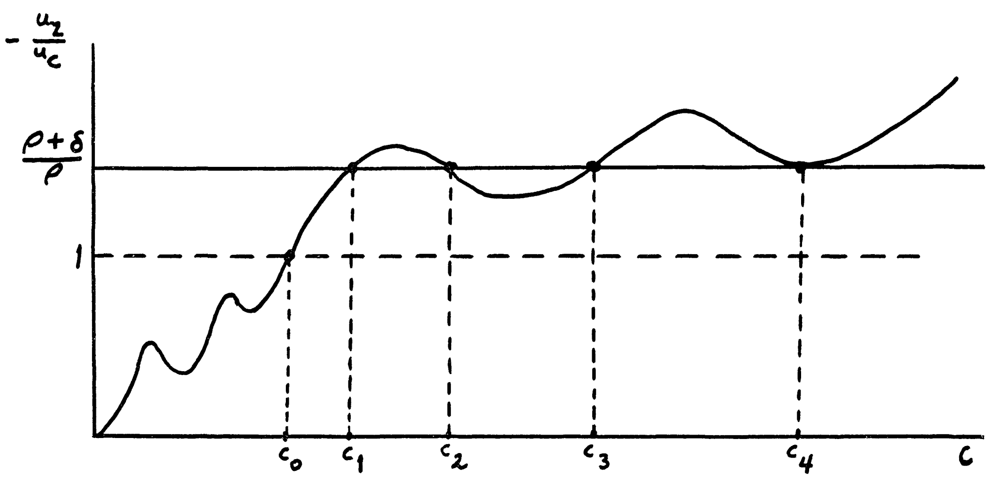

FIGURE 6

The effect of changes in δ can be seen in Figure 6 which shows the slopes of the indifference curves along the locus $c = z$. By concavity, this curve can cross $-u_z/u_c = 1$ only once at c_0, but nothing guarantees its monotonicity at other points. It is easily shown that $q \gtreqless 0$ as $-u_z/u_c \lesseqgtr (\rho+\delta)/\rho$. Satiated optimal stationary solutions occur where the curve crosses $-u_z/u_c = (\rho+\delta)/\rho$. If there is no discounting, $\delta = 0$, there is a unique satiated optimal stationary solution at c_0. An increase in δ will increase solutions like c_1 and c_3

where $q'(c) < 0$, and will decrease solutions like c_2 where $q'(c) > 0$. A double solution like c_4 will separate into two distinct solutions if δ is increased and will vanish if δ is decreased.

Figure 7 shows how $c = z$ varies with k across stationary states.
By (4) and (6), if $\dot{k} = \dot{z} = 0$, we must have

$$c = z = f(k) - \lambda k.$$

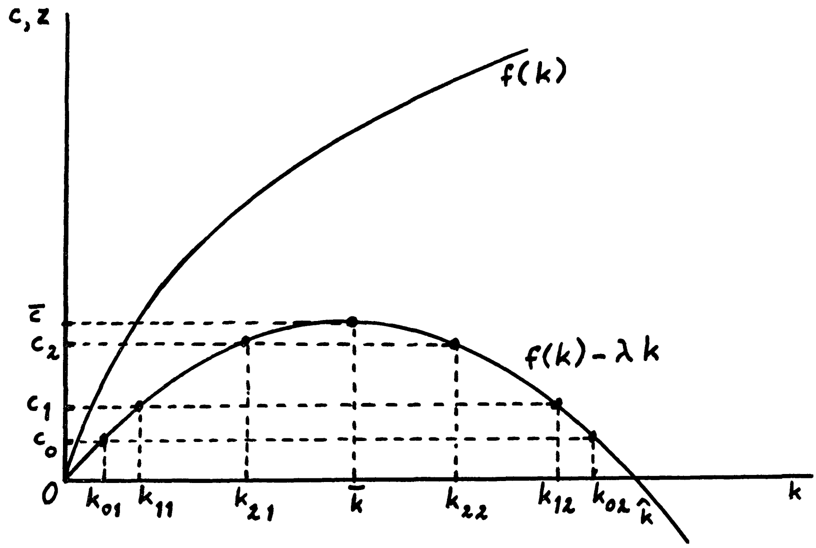

FIGURE 7

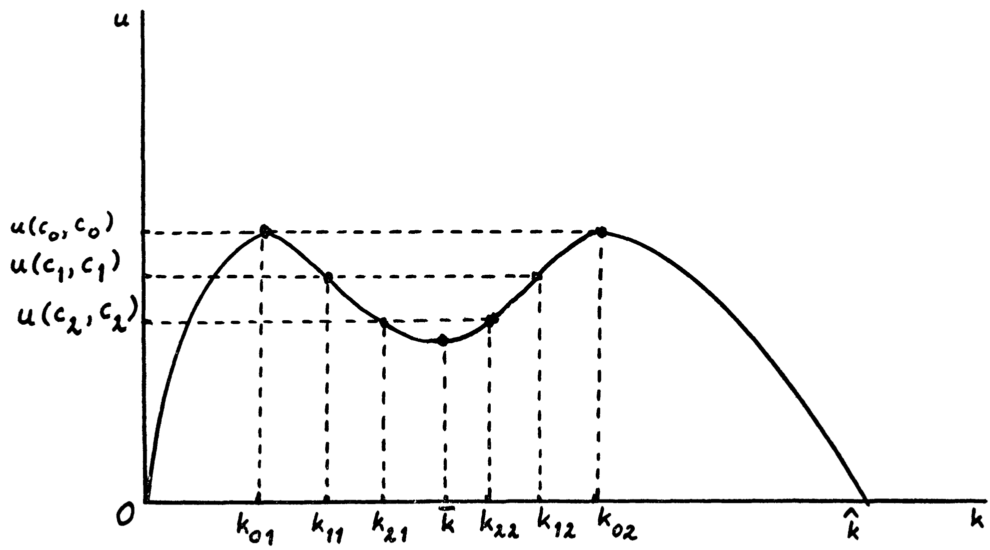

FIGURE 8

As k rises from 0 through $\bar{k}$ to $\hat{k}$, c rises from 0 to $\bar{c}$ at $\bar{k}$ (the golden rule point) and then falls back to 0. If $c_1 > \bar{c}$, the existence of satiation in the utility function is irrelevant, since it is not feasible to sustain a satiated optimal stationary solution. If, as shown in Figure 7, we have one or more $c_i < \bar{c}$, then each corresponds to two possible levels of k: k_{i1} and k_{i2} where

$$0 < k_{i1} < \bar{k} < k_{i2} < \hat{k}.$$

The relationship between $u(c, c)$ and k across stationary states is given in Figure 8. There are two maxima at k_{01} and k_{02} and a local minimum at the golden rule $\bar{k}$. Satiated optimal stationary solutions lie between k_{01} and k_{02}. They generally come in pairs and are arranged more or less symmetrically around $\bar{k}$. With no discounting, $\delta = 0$, there would be only $k_{11} = k_{01}$ and $k_{12} = k_{02}$. As δ is increased c_1 is increased so k_{1j} moves away from k_{0j} toward $\bar{k}$; as c_2 is decreased so k_{2j} moves away from $\bar{k}$ toward k_{0j}.

5. CHARACTERIZATION OF OPTIMAL PATHS

The following sections are concerned with the nature of optimal paths in the model described above. These paths are found to differ from those described for the one-sector model by Cass [2], Mirrlees [13], Ramsey [16] and others. There is still a unique optimal stationary solution (the modified " golden rule "), but the optimal trajectory may overshoot this target. This overshooting may be repeated and need not converge to the optimal stationary solution. When we relax assumption (P. 3) and allow satiation, there may be a multiplicity of optimal stationary solutions. Kurz [11], working with a model where the capital stock appeared as an argument of the welfare function, also found a multiplicity of stationary solutions. This similarity is not altogether surprising, as there is some similarity in the formal structures of the two models. The paper by Chakravarty and Manne [4], in which the argument of the instantaneous welfare function is the rate of change of the level of per capita consumption, seems to be concerned with a problem similar to the present one: but the formulation, and therefore the results, are very different.

Samuelson [18] and Wan [21] have examined discrete-time models closely related to ours. Samuelson found turnpike properties when δ is " small ". We find that counter-examples are possible when δ is " large ". The most striking results of Wan's explorations involve satiated paths with interdependence within the planning period.

In problems of this sort, the existence of an optimal solution is usually established (implicitly) by a round-about process—namely one establishes necessary and sufficient conditions for optimality and constructs a path which satisfies these conditions. Because of the difficulty of the construction in this model, we must attack the existence problem directly.

Theorem 1. *Under assumptions (P. 4), (T. 1) and (T. 2) there exists a unique consumption path that maximizes (3) subject to (4), (5), (6), (7). The proof is given in Appendix A.*

In order to characterize the optimal path we shall apply the familiar Maximum Principle of Pontryagin [15]. Suppose there are functions $p(t)$ and $q(t)$, $t \geqq 0$, representing the shadow prices of z and k respectively. Let us define the Hamiltonian as

$$H(p, q, z, k, c) \equiv u(c, z) + p\rho(c-z) + q[f(k) - \lambda k - c]. \qquad ...(23)$$

$$M(p, q, z, k) \equiv \max_{0 \leqq c \leqq f(k)} H(p, q, z, k, c). \qquad ...(24)$$

The conditions of optimality are now

$$H(p, q, z, k, c) = M(p, q, z, k), \qquad ...(25)$$

$$\dot{p} = \delta p - \frac{\partial M}{\partial z}, \qquad ...(26)$$

$$\dot{q} = \delta q - \frac{\partial M}{\partial k}, \qquad ...(27)$$

$$p(t) \leqq 0 \text{ for all } t \qquad ...(28)$$

$$q(t) \geqq 0 \text{ for all } t \qquad ...(29)$$

$$\lim_{t\to\infty} e^{-\delta t}p(t)z(t) = 0 \qquad \text{...(30)}$$

$$\lim_{t\to\infty} e^{-\delta t}q(t)k(t) = 0. \qquad \text{...(31)}$$

Theorem 2. *Suppose there exist functions* $p(t)$, $q(t)$, $z^0(t)$, $k^0(t)$, $c^0(t)$ *satisfying* (4)-(7), (25)-(31). *Then* $\{z^0(t), k^0(t), c^0(t)\}$ *is the optimal path.*

This theorem is a straightforward application of a familiar result, since the maximized Hamiltonian is concave in the state variables. See, for example, Arrow and Kurz [1; Section II. 6].

If future utility is not discounted, $\delta = 0$, the functional (3) diverges. We may still obtain an ordering of feasible consumption path using the overtaking criterion of Von Weizsäcker [22]. We replace the transversality conditions (30), (31) in Theorem 2 by

$$\lim_{t\to\infty} p(t)z(t) < \infty \qquad \text{...(32)}$$

$$\lim_{t\to\infty} q(t)k(t) < \infty. \qquad \text{...(33)}$$

The proof, a straightforward adaptation of Von Weizsäcker's proof, is omitted.

Condition (29) is of some importance in distinguishing between paths, and can be justified by the following argument. Suppose $k(t)$ and $c(t)$ to form an optimal path starting from $k(0)$ and $z(0)$. $k(0)$ is raised to $k'(0)$ and a new consumption path is defined by $c'(t) = c(t)$. This consumption policy is clearly feasible in a right-hand neighbourhood of $t = 0$, and is in fact perpetually feasible. For let $k'(t)$ be the resulting time-path of the capital stock: if $k'(t) > k(t)$ for all t, then the new policy is clearly feasible and has the same payoff as the old one. If $k'(t_0) = k(t_0)$ for some t_0, then the original consumption policy is again feasible as it is possible to set $k'(t) = k(t)$ for all $t > t_0$. Additional initial capital can, therefore, never reduce the payoff associated with an optimal policy. But $q(0)$ is the increment in payoff associated with an increment in k_0, so this establishes that $q(0) \geqq 0$ on an optimal path. (See, for example, Peterson [14].) Obviously the same argument can be used for any $t > 0$, thus establishing the point.

Let us examine the optimal path more closely. There are two possible phases. In Phase I, condition (25) has a corner solution:

$$c = f(k), \qquad \text{...(5-1)}$$

$$u_c > q - \rho p. \qquad \text{...(25-1)}$$

In Phase II, it has an interior solution:

$$0 < c \leqq f(k), \qquad \text{...(5-2)}$$

$$u_c = q - \rho p. \qquad \text{...(25-2)}$$

Assumption (P. 5) implies that it can never be optimal to consume nothing. The equations describing the behaviour of the system are

$$\dot{k} = -\lambda k, \qquad \text{Phase I} \qquad \text{...(4-1)}$$

$$\dot{k} = f(k) - \lambda k - c, \qquad \text{Phase II} \qquad \text{...(4-2)}$$

$$\dot{z} = \rho[f(k) - z], \qquad \text{Phase I} \qquad \text{...(6-1)}$$

$$\dot{z} = \rho(c - z), \qquad \text{Phase II} \qquad \text{...(6-2)}$$

$$\dot{p} = (\delta + \rho)p - u_z, \qquad \text{...(26-1, 2)}$$

$$\dot{q} = (\delta + \lambda)q - (u_c + \rho p)f'(k), \qquad \text{Phase I} \qquad \text{...(27-1)}$$

$$\dot{q} = [\delta + \lambda - f'(k)]q. \qquad \text{Phase II} \qquad \text{...(27-2)}$$

The description of the optimal paths is best conducted by locating stationary solutions to the system (4), (6), (26) and (27), and then discussing the manner in which such points may be approached. It is easy to verify that there is no stationary solution in Phase I, for in this phase,

$$\dot{k} = \dot{z} = 0 \text{ iff } k = z = 0. \quad \text{However, } \dot{q} = 0 \text{ iff } q = \frac{(u_c + \rho p) f'(k)}{\delta + \lambda} < u_c + \rho p$$

which can occur only if $f' < (\delta + \lambda)$: but by assumption (T. 2) this is impossible when $k = 0$. In contrast to this, Phase II may be richly endowed with stationary solutions. These fall into two categories.

(i) One stationary point is the conventional modified golden rule solution. This is given by

$$f'(k^*) = \lambda + \delta \qquad \ldots(34)$$

$$c^* = f(k^*) - \lambda k^* \qquad \ldots(35)$$

$$z^* = c^* \qquad \ldots(36)$$

$$u_z(c^*, z^*) = (\delta + \rho) p^* \qquad \ldots(37)$$

$$q^* = u_c(c^*, z^*) + \frac{\rho}{\delta + \rho} u_z(c^*, z^*). \qquad \ldots(38)$$

Under assumption (P. 3),

$$q^* > \frac{\rho}{\delta + \rho} (u_c + u_z) > 0.$$

(ii) The second category of stationary solutions, which must be empty unless we relax assumption (P. 3), consists of the satiated optimal stationary solutions described above in Section 4. These solutions satisfy

$$q = u_c(c_i, c_i) + \frac{\rho}{\rho + \delta} u_z(c_i, c_i) = 0 \qquad \ldots(39)$$

$$z_i = c_i \qquad \ldots(40)$$

$$p_i = \frac{1}{\delta + \rho} u_z(c_i, c_i) \qquad \ldots(41)$$

$$f(k_{ij}) - \lambda k_{ij} = c_i \quad j = 1, 2. \qquad \ldots(42)$$

In the neighbourhood of a stationary point, the system can be approximated by a linear system. From (25-2) we obtain

$$c - c^* = -\frac{u_{cz}}{u_{cc}}(z - z^*) - \frac{\rho}{u_{cc}}(p - p^*) + \frac{1}{u_{cc}}(q - q^*). \qquad \ldots(43)$$

Then

$$\begin{bmatrix} \dot{z} \\ \dot{k} \\ \dot{p} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -\rho\left(1 + \frac{u_{cz}}{u_{cc}}\right) & 0 & -\frac{\rho^2}{u_{cc}} & \frac{\rho}{u_{cc}} \\ \frac{u_{cz}}{u_{cc}} & f' - \lambda & \frac{\rho}{u_{cc}} & -\frac{1}{u_{cc}} \\ u_{zz} + \frac{(u_{cz})^2}{u_{cc}} & 0 & \delta + \rho\left(1 + \frac{u_{cz}}{u_{cc}}\right) & -\frac{u_{cz}}{u_{cc}} \\ 0 & -q^* f'' & 0 & \delta + \lambda - f' \end{bmatrix} \begin{bmatrix} z - z^* \\ k - k^* \\ p - p^* \\ q - q^* \end{bmatrix} \qquad \ldots(44)$$

If μ is a characteristic root of this dynamic system, we have

$$(\delta+\lambda-f'-\mu)(f'-\lambda-\mu)[\mu^2-\delta\mu-\rho\gamma-\rho(\delta+\rho)]-\beta[\mu^2-\delta\rho\mu-\rho(\delta+\rho)] = 0 \quad \ldots(45)$$

where

$$\beta = \frac{qf''}{u_{cc}} \quad \ldots(46)$$

$$\gamma = (2\rho+\delta)\frac{u_{cz}}{u_{cc}}+\rho\frac{u_{zz}}{u_{cc}} \quad \ldots(47)$$

and all derivatives are evaluated at the stationary solution in question. Note that $\gamma<0$ under adjacent complementarity, $\gamma = 0$ under intertemporal independence, and $\gamma>0$ under distant complementarity. (See Section 3.)

We will first describe the motion about the modified golden rule solution. We will then consider satiated optimal stationary solutions.

6. THE MODIFIED GOLDEN RULE SOLUTION

Under Assumption (P. 3) the modified golden rule solution is the only stationary solution of the system. By (19), (38) we have

$$q^* = q(c^*)>0, \beta>0$$

$$\delta+\lambda-f'(k^*) = 0.$$

Then the characteristic equation (45) simplifies to

$$\mu^4-2\delta\mu^3+(\delta^2-\sigma)\mu^2+\delta\sigma\mu+\tau = 0 \quad \ldots(48)$$

where

$$\sigma = \rho[\rho+\delta+\gamma]+\beta \quad \ldots(49)$$

$$\tau = \rho(\rho+\delta)\beta>0. \quad \ldots(50)$$

The four roots of this polynomial are given by

$$\mu = \frac{\delta}{2}\pm\tfrac{1}{2}\sqrt{\frac{\delta^2}{2}+\sigma+\sqrt{\frac{\delta^4}{4}+\delta^2\sigma+4\tau}}\pm\tfrac{1}{2}\sqrt{\frac{\delta^2}{2}+\sigma-\sqrt{\frac{\delta^4}{4}+\delta^2\sigma+4\tau}}. \quad \ldots(51)$$

By the assumption of concavity (P. 4) we have

$$\underline{\gamma} \equiv -\frac{1}{\rho}\left(\frac{\delta}{2}+\rho\right)^2 \leqq \gamma, \quad \ldots(52)$$

$$\beta-\frac{\delta^2}{4} \leqq \sigma. \quad \ldots(53)$$

The first radical in (51) is always real; the second may be imaginary or real. The roots are symmetrical about $\delta/2$. For $\delta = 0$, there must, therefore, be two stable roots (negative real parts) and two unstable roots (positive real parts). By continuity, this must also hold when $\delta>0$ is small. We shall see presently what may happen when δ is large.

Assuming for the present that we have two stable and two unstable roots, we may assure convergence to the modified golden rule by restricting ourselves to the plane spanned by the characteristic vectors corresponding to the two stable roots. If all four roots have positive real parts, the linear approximation can still be derived by examining the motion associated with the two roots having the smallest real parts. These roots are found by choosing the $(-)$ sign for the first radical in equation (51). In this way we obtain a decision rule for the control variable $c = c(z, k)$. At the modified golden rule, we have $c(z^*, k^*) = c^*$. We can show (see Appendix B) that

$$c_k(z^*, k^*) > 0, \qquad \ldots(54)$$

and

$$c_z(z^*, k^*) \gtreqless 0 \text{ as } \gamma \lesseqgtr 0,\ c_z(z^*, k^*) < 1. \qquad \ldots(55)$$

Thus an increase in capital will increase optimal current consumption. Under adjacent complementarity, an increment in the customary level of consumption will result in a lesser increment in the same direction to the optimal level of current consumption. Under intertemporal independence, an increment in customary consumption has no effect on the optimal current consumption. Under distant complementarity, an increment in customary consumption will result in an increment in the opposite direction to optimal current consumption.

Now let us examine the patterns of optimal motion about the stationary point. This motion will converge to the modified golden rule if we have two stable roots. It will have the form of a node if these roots are real, a focus (spiral) if they are complex. The product of all four roots of the polynomial (48) is

$$\tau > 0. \qquad \ldots(56)$$

None of the roots can vanish. Thus ambiguity about the stability of the roots (51) can occur only in case they are pure imaginary.

Spiralling occurs if

$$\sqrt{\frac{\delta^4}{4} + \delta^2\sigma + 4\tau} > \frac{\delta^2}{2} + \sigma$$

or equivalently,

$$\gamma_3 \equiv \frac{-[\sqrt{\rho^2 + \rho\delta} + \sqrt{\beta}]^2}{\rho} < \gamma < \frac{-[\sqrt{\rho^2 + \rho\delta} - \sqrt{\beta}]^2}{\rho} \equiv \gamma_1 \leqq 0, \qquad \ldots(57)$$

The motion is stable if

$$\sqrt{\frac{\delta^2}{2} + \sigma + \sqrt{\frac{\delta^4}{4} + \delta^2\sigma + 4\tau}} > \delta,$$

or equivalently

$$\gamma > (\delta^2 - (\rho^2 + \rho\delta + \beta) - \sqrt{\delta^4 + 4(\rho + \rho\delta)\beta})/\rho \equiv \gamma_2. \qquad \ldots(58)$$

The γ_1, γ_2 and γ_3 defined in (57) and (58) satisfy $0 \geqq \gamma_1 > \gamma_2 > \gamma_3$.

Under intertemporal independence or distant complementarity, we always get a stable node. $\underline{\gamma}$, the minimum value of γ consistent with concave utility, was defined in (52). A little manipulation shows that

$$\gamma_1 \gtreqless \underline{\gamma} \text{ if } \delta \gtreqless 2\sqrt{\beta} - 2\sqrt{2\rho\sqrt{\beta}},$$

$$\gamma_2 \gtreqless \underline{\gamma} \text{ if } \delta \gtreqless \tfrac{2}{3}[\sqrt{\beta} + 2\sqrt{\beta + \tfrac{3}{2}\rho\sqrt{\beta}}] > 2\sqrt{\beta},$$

$$\gamma_3 \gtreqless \underline{\gamma} \text{ if } \delta \gtreqless 2\sqrt{\beta} + 2\sqrt{2\rho\sqrt{\beta}}.$$

Thus under weak discounting, the motion must be stable, but under sufficiently strong discounting and appropriately weak concavity or utility, instability may occur. Figure 9 shows the regions of stability for various admissible values of γ and δ, given the values of β and ρ.

In the proof of Theorem 1, we have shown that z and k are uniformly bounded on all feasible paths. We have shown that under assumption (P. 3) the modified golden rule (34), (35), (36), (37), (38) is the only stationary solution with finite values for p and q. We have seen that this stationary solution is stable if either discounting or interdependence is weak. If interdependence is very weak, we will have a stable node as illustrated in Figure 10. With

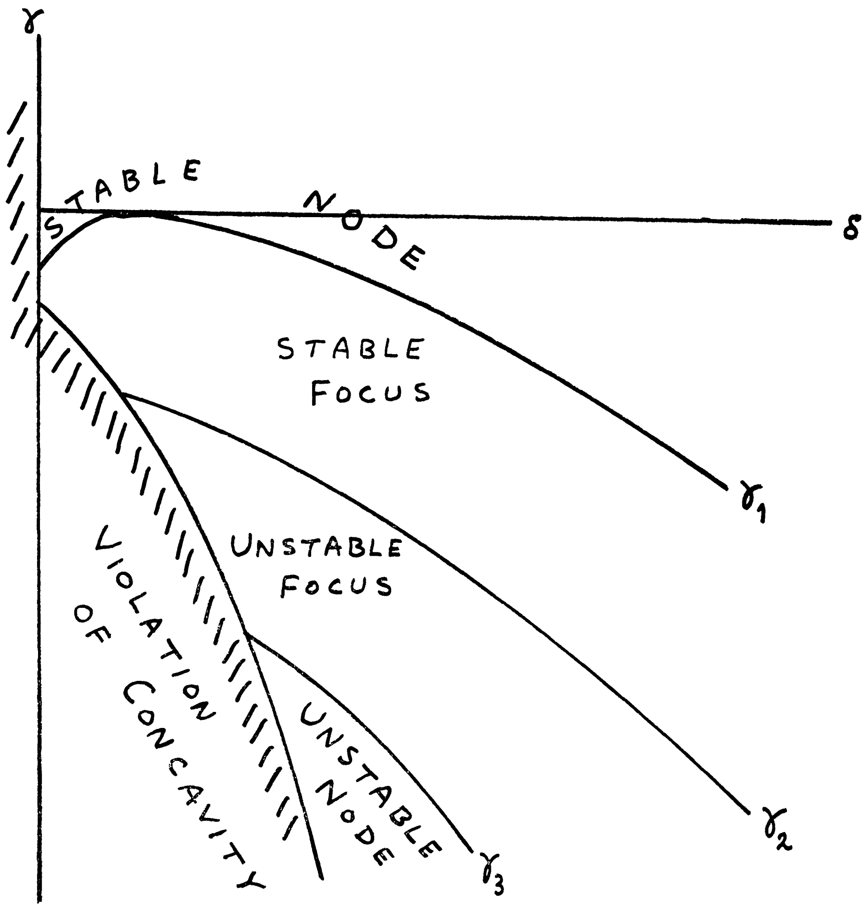

FIGURE 9

Regions of stability.

somewhat stronger interdependence, we will have a stable focus as illustrated in Figure 11.

When both discounting and interdependence are strong, the stationary point may be unstable as illustrated in Figure 12. In this case, unless we start from the stationary point, we can never get there on an optimal path. But then, where else can we go? There are two other possible stationary points in z and k, but to approach them entails asymptotically diverging values of p and q. These points are $z = 0, k = \hat{k}$ and $z = 0, k = 0$. The first of these cannot be optimal, since it can be dominated by the usual "golden rule" argument. The second cannot be optimal since q must eventually become negative as the origin is approached.

There is, however, a third possible asymptotic behaviour. That is to approach no final state of rest, but rather to oscillate endlessly in a stable limit cycle. This is the outcome shown in Figure 12.

Let us examine the motion along a typical optimal path. A country with a low capital stock but a high level of recent consumption might start by reducing its consumption level gradually, so that capital may continue to fall for a while. Soon consumption will have been reduced to the level which permits capital to start accumulating again. The process of belt-tightening will slow down and eventually reverse as more capital is accumulated. As capital grows, it will reach the modified golden rule level, but by now the economy is so used to low consumption, that there is no rush to raise it to the long-run optimal level.

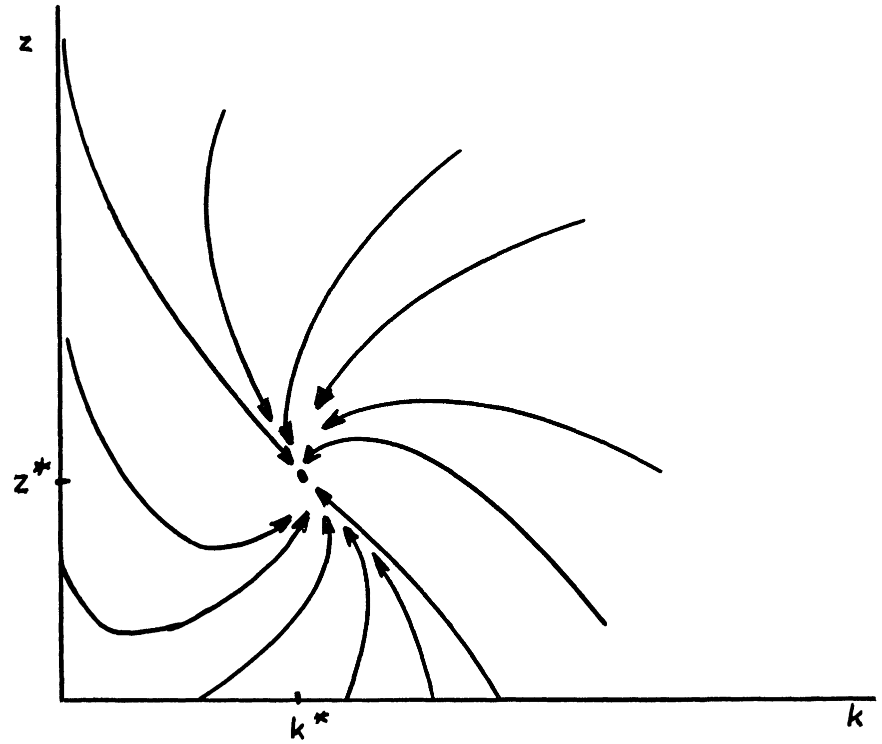

FIGURE 10
Stable node $\gamma_1 \leqq \gamma \leqq 0$.

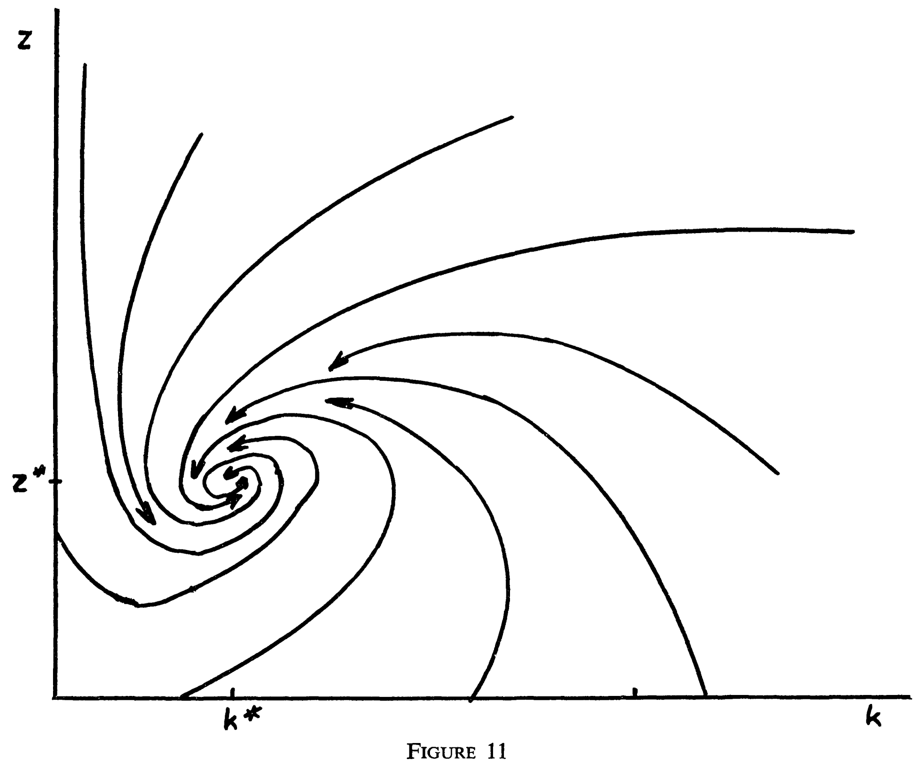

FIGURE 11
Stable focus $\gamma_2 < \gamma < \gamma_1$.

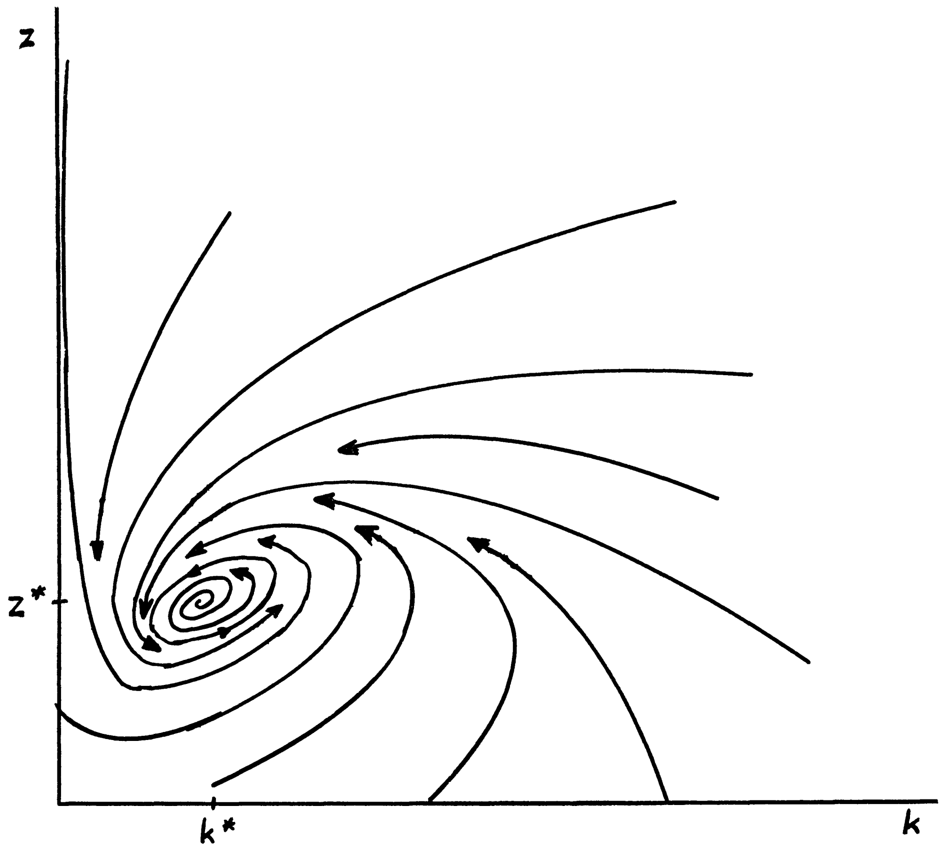

FIGURE 12

Unstable focus with limit cycle $\gamma_3 < \gamma < \gamma_2$.

Instead, consumption will rise gradually, eventually reaching the level that causes capital to stop rising and begin falling. As capital falls the rise in consumption will slow down and eventually reverse as we come once more to the condition of a high level of consumption and a low capital stock.

Depending on the parameters of the model, this overshooting may occur only once, or it may be repeated over and over again. In the latter case, the overshooting may be by a lesser amount on each iteration as the economy narrows down on its target. Or it may wind up repeating the same moves over and over without ever getting closer to the stationary point.

7. SATIATED OPTIMAL STATIONARY SOLUTIONS

If we relax assumption (P. 3) we may find one or more satiated optimal stationary solutions as discussed above in Section 4. Recall from Section 4 our definition

$$q(c) \equiv u_c(c, c) + \frac{\rho}{\delta+\rho} u_z(c, c). \qquad \ldots(19)$$

If $q(c^*) > 0$ where c^* is the consumption at the modified golden rule, then the modified golden rule is an optimal stationary solution and has the local properties worked out in Section 6 above. This would occur if $c^* < c_1$ or if $c_2 < c^* < c_3$ for the configuration of

Figure 5. If $q(c^*)<0$, the modified golden rule is not an optimal stationary solution, since it violates condition (29). This would occur if $c_1<c^*<c_2$ or $c^*>c_3$ in Figure 5.

At a satiated optimal stationary solution, the linearized dynamic system (44) is simplified since $q = \beta = 0$.

The characteristic quartic equation (45) now takes the simple form

$$(\delta+\lambda-f'-\mu)(f'-\lambda-\mu)[\mu^2-\delta\mu-\rho(\rho+\delta+\gamma)] = 0. \qquad \text{...(59)}$$

In the neighbourhood of a satiated optimal stationary solution, the capital constraint irrelevant. Let us, therefore, neglect it and solve for the paths satisfying

$$u_c = -\rho p \qquad \text{...(60)}$$

$$\dot{z} = \rho(c-z) \qquad \text{...(61)}$$

$$\dot{p} = (\delta+\rho)p-u_z. \qquad \text{...(62)}$$

The characteristic equation is

$$\mu^2-\delta\mu-\rho(\rho+\delta+\gamma) = 0. \qquad \text{...(63)}$$

The roots are

$$\mu = \frac{\delta}{2} \pm\tfrac{1}{2}\sqrt{\delta^2+4\rho(\rho+\delta+\gamma)}. \qquad \text{...(64)}$$

By (21) and (47) the roots are real. By (20) and (47), we see that both are positive when $q'(c_i)>0$ and that the signs are opposite when $q'(c_i)<0$. The phase diagram is shown in

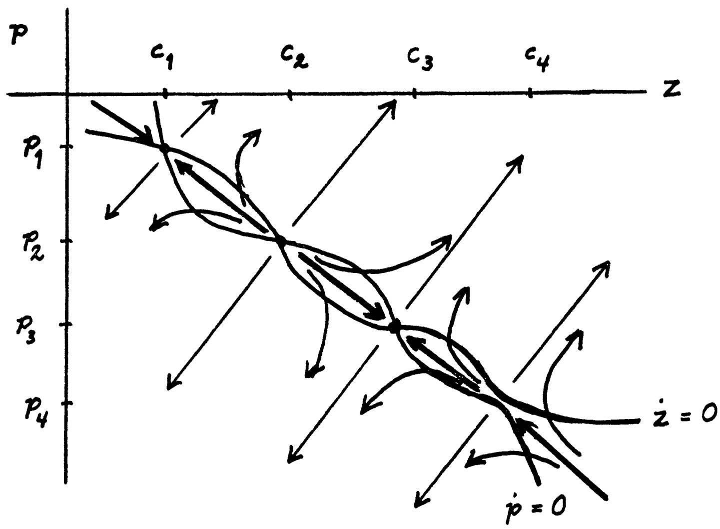

FIGURE 13

Figure 13. Note in particular that the first solution c_1 is a saddle point, and that successive solutions alternate between saddle points and unstable nodes (a tangency solution such as c_4 is counted twice). The optimal paths (heavy arrows) move away from the adjacent node toward the adjacent saddle point. In particular, if z is small, we approach c_1.

Now we can simply add k and q to the system by setting

$$\dot{k} = f(k)-\lambda k-c \qquad \text{...(65)}$$

$$\dot{q} = q = 0. \qquad \text{...(66)}$$

As long as $c \leqq f(k)$, we remain in Phase II. Then the system is feasible, and paths tending to stationary solutions satisfy Theorem 2.

At points where $c = f(k)$ we have the boundary of Phase II. If the motion is crossing the boundary into Phase II, we may extend the solution backward into Phase I using equations (4-1)-(6-1), (25-1)-(27-1). If the motion is crossing out of Phase II, we may extend the

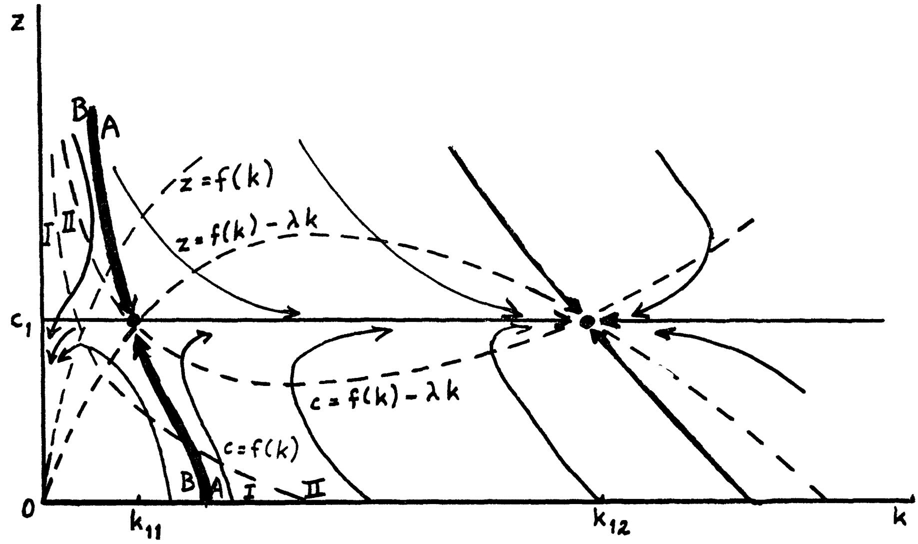

FIGURE 14

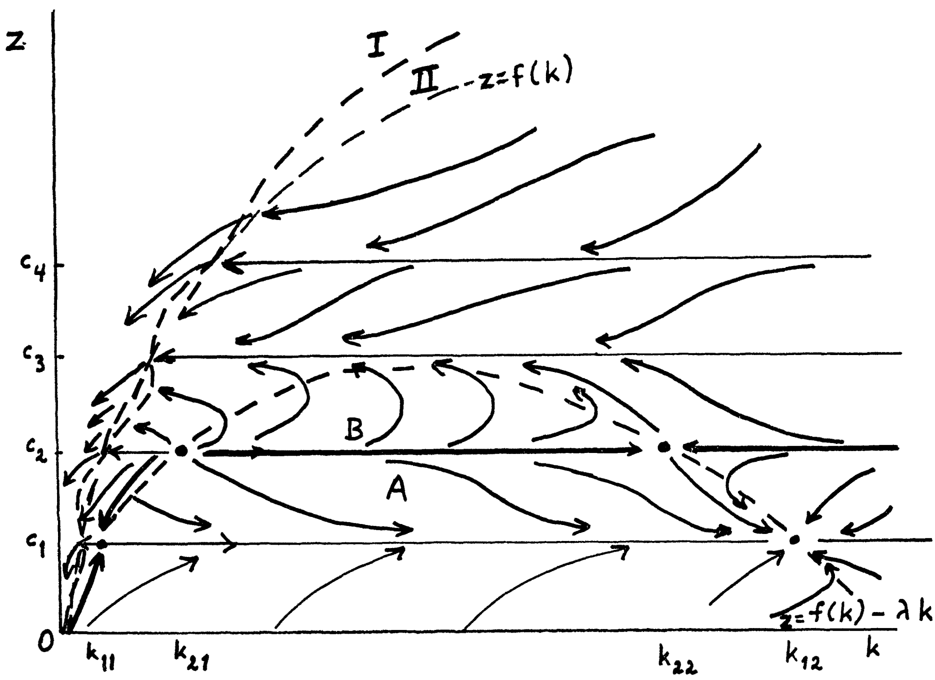

FIGURE 15

solution forward into Phase I. Figure 14 shows the configuration under distant complementarity with a unique satiated optimal stationary solution c_1. Figure 15 shows a more complicated configuration, possible under adjacent complementarity, with a multiplicity of satiated optimal stationary solutions. In either case there are certain critical paths (shown as heavy arrows) which divide the positive quadrant into regions A and B. In region A, all motion converges to some proper stationary point in Phase II. In region B, all motion ultimately enters Phase I, where q immediately becomes negative, and z and k approach the origin, where there is weeping and wailing and gnashing of teeth by assumption (P. 5). The boundary of region A will pass through one or more stationary points, and wherever it does, will be tangent to one of the characteristic vectors of that stationary point. The characteristic values are

$$\begin{aligned} \mu_1 &= f' - \lambda, \\ \mu_2 &= \delta + \lambda - f', \\ \mu_3 &= \frac{\delta}{2} + \tfrac{1}{2}\sqrt{\delta^2 + 4\rho(\rho + \delta + \gamma)}, \\ \mu_4 &= \frac{\delta}{2} - \tfrac{1}{2}\sqrt{\delta^2 + 4\rho(\rho + \delta + \gamma)}. \end{aligned} \qquad \ldots(67)$$

μ_1 is stable for $k > \bar{k}$ and unstable for $k < \bar{k}$. μ_2 is stable for $k < k^*$ and unstable for $k > k^*$. We have already discussed the stability of μ_3 and μ_4. The motion described in Figures 14 and 15 corresponds to μ_1 and μ_4.

Let $\{z^h, k^h, p^h, q^h\}$ be the characteristic vector corresponding to μ_h. Then

$$(\rho + \mu_h) z^h = \rho(f' - \lambda - \mu_h) k^h, \qquad \ldots(68)$$

$$\rho(\rho + \delta - \mu_h) q^h = -u_{cc}[\mu_h^2 - \delta\mu_h - \rho(\rho + \delta + \gamma)] z^h. \qquad \ldots(69)$$

As in Section 6 above, there is an optimal decision rule $c = c(z, k)$. In region A, this rule depends only on z. In region B, its derivatives may be computed by examining the linear motion in the neighbourhood of the stationary points. This motion will be associated with the two smallest characteristic roots, unless such motion violates an optimality condition. For large values of $k, f' - \lambda < \delta/2$, we have $\mu_1 < \mu_2$, so the two smallest roots are μ_1 and μ_4. For small values of $k, f' - \lambda > \delta/2$, we have $\mu_2 < \mu_1$, so the two smallest roots are μ_2 and μ_4. Substituting (67) into (68) and (69) yields $z^1 = q^1 = q^3 = q^4 = 0$. But in general $q^2 \neq 0$. For $\mu_2 < \mu_1$, we can show that $q^2 < 0$ to the right of the characteristic vector (z^4, k^4) and $q^2 > 0$ to the left. Thus motion associated with μ_2 violates condition (29) to the right of (z^4, k^4), so in this region, the linear approximation of the optimal motion is associated with μ_1 and μ_4. We shall call the partial derivatives in this region c_k^A and z_z^A, since the region includes all of region A, where capital is eternally abundant, as well as those parts of region B in which capital is initially abundant.

To the left of (z^4, k^4) at any satiated stationary point with $f' - \lambda > \delta/2$, we have the region in which optimal paths economize capital right from the beginning. Since this regime is a subset of region B, the partial derivatives of the decision function in this region will be called c_k^B and c_z^B. We show in Appendix B that

$$c_k^A(c_i, k_{ij}) = 0, \qquad \ldots(70)$$

$$c_k^B(c_i, k_{i1}) > 0, \qquad \ldots(71)$$

$$c_z^A(c_i, k_{ij}) \gtreqless 1 \text{ as } q'(c_i) \gtreqless 0, \qquad \ldots(72)$$

$$c_z^A(c_i, k_{ij}) \gtreqless 0 \text{ as } \gamma \lesseqgtr 0, \qquad \ldots(73)$$

$$\frac{c_z^B(c_i, k_{i1})}{c_z^A(c_i, k_{i1})} < 1, \qquad \ldots(74)$$

$$\frac{c_z^B(c_i, k_{i1})}{c_z^A(c_i, k_{i1})} \gtreqless 0 \text{ as } f' - \lambda \lesseqgtr \delta + \rho. \qquad \ldots(75)$$

By (70) and (71) we see that changes in the capital stock have no effect on consumption in the region of abundance, but a reduction of capital in the region of capital shortage will reduce consumption. If capital is abundant, the effects of an increment in customary consumption are clear. Under distant complementarity, optimal current consumption will move in the opposite direction. Under adjacent complementarity current consumption will move in the same direction by a lesser amount at a stable satiation point and by a greater amount at an unstable satiation point. These effects are diminished in the scarcity region and are even reversed if the marginal product of capital is high. In the latter case $c(z, k_{i1})$ has a local maximum at $z = c_i$. If customary consumption is incremented in one direction (depending on the pattern of complementarity), current consumption is reduced for purely preferential reasons, moving us into the region of abundance. If customary consumption is incremented in the other direction, the preferred increase in

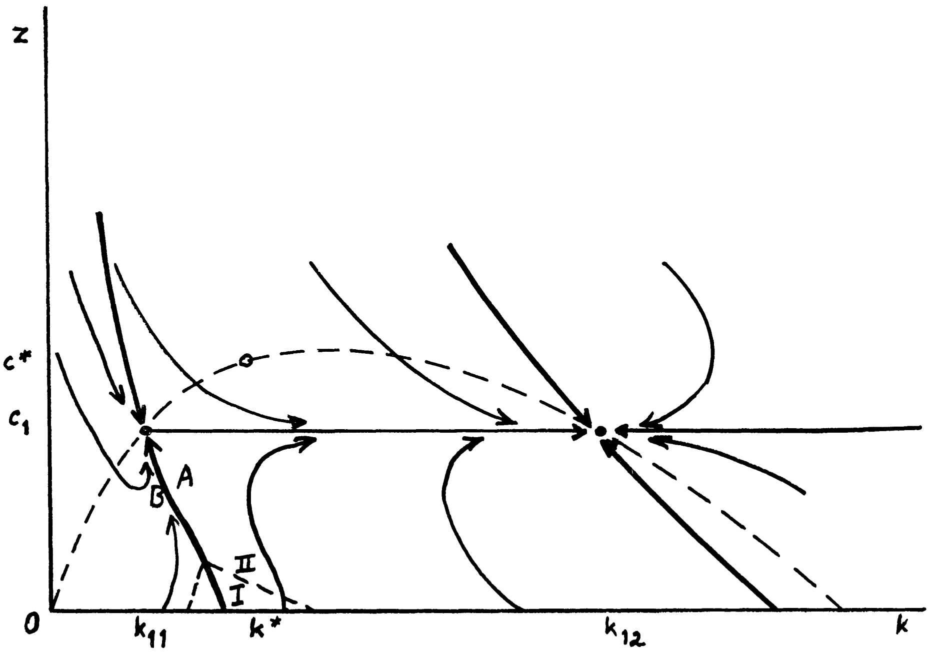

FIGURE 16

current consumption would lead to a capital shortage. But since the marginal product of capital is so high, it is better to reduce consumption temporarily, permitting a much larger increase in consumption after the capital stock has grown.

We turn now to the task of piecing together all these fractions of optimal paths. For the configuration shown in Figure 14, we may consider two cases. If $c_1 < c^*$, then $q^* = q(c^*) < 0$. The stationary point (k^*, z^*) lies inside region A and is dominated by satiation solutions. The phase diagram for this case is given in Figure 16. If $c^* < c_1, q^* = q(c^*) > 0$ and (k^*, z^*) is an optimal stationary solution in region B. For the configuration shown in Figure 15, there are three cases (Figures 17, 18 and 19). The modified golden rule is an optimal stationary point if $c^* < c_1$ (Figure 17) or if $c_1 < c_2 < c^*$ (Figure 19). But for $c_1 < c^* < c_2$ (Figure 18), the modified golden rule is dominated by satiated solutions.

8. NUMERICAL RESULTS

The general analysis conducted above has revealed a number of interesting possible configurations for optimal paths. In this section we compute the roots of the quartic (45) for

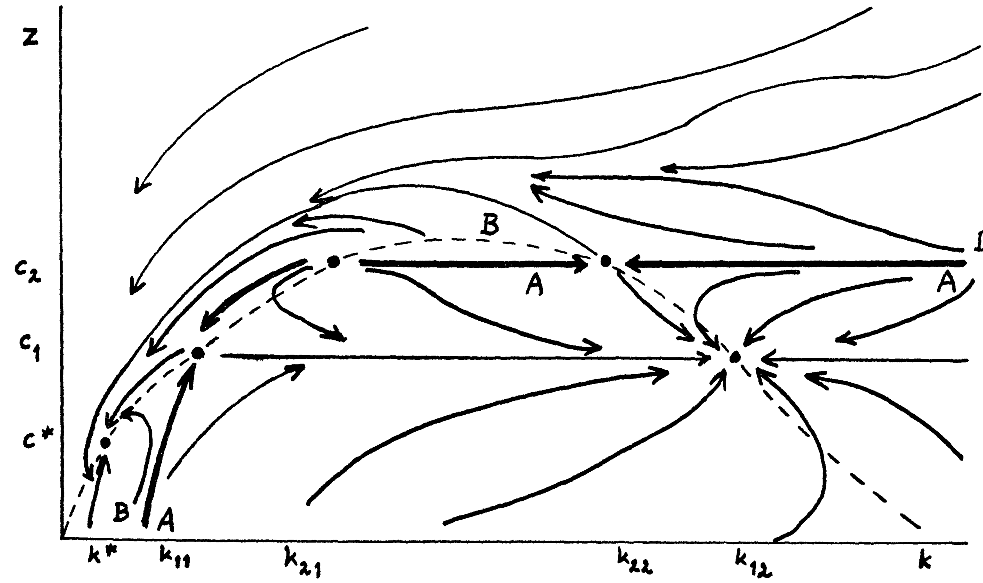

FIGURE 17

$c^* < c_1$.

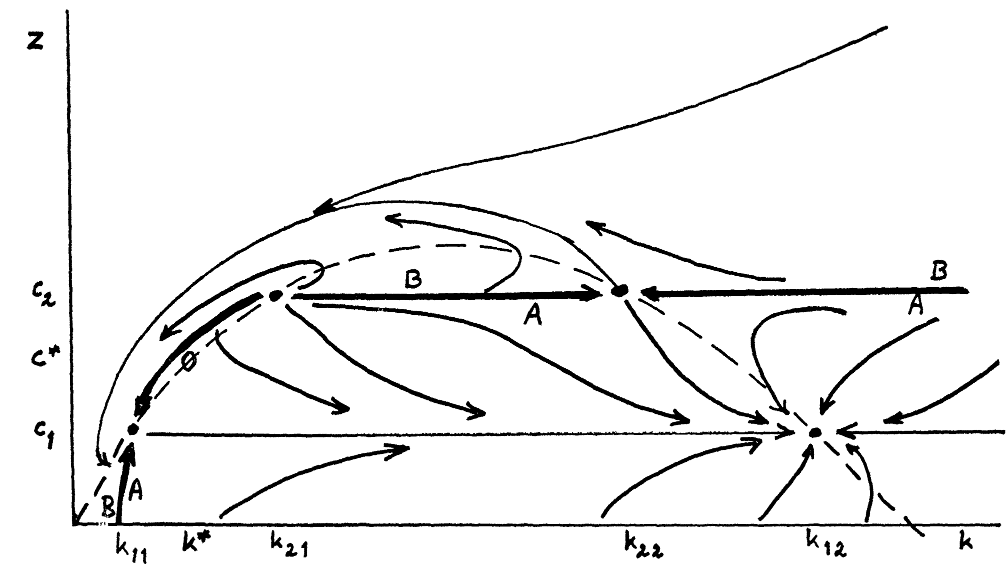

FIGURE 18

$c_1 < c^* < c_2$.

a variety of plausible functional forms, and study the behaviour of optimal paths in neighbourhoods of stationary solutions.

In all calculations, a Cobb-Douglas production function of the form $f(k) = k^{\frac{1}{3}}$ was used. Values had to be assigned to the three parameters δ, λ and ρ: in the calculations reported, δ and λ were set equal to 0·01 (in a number of trial cases, δ was varied between 0·01

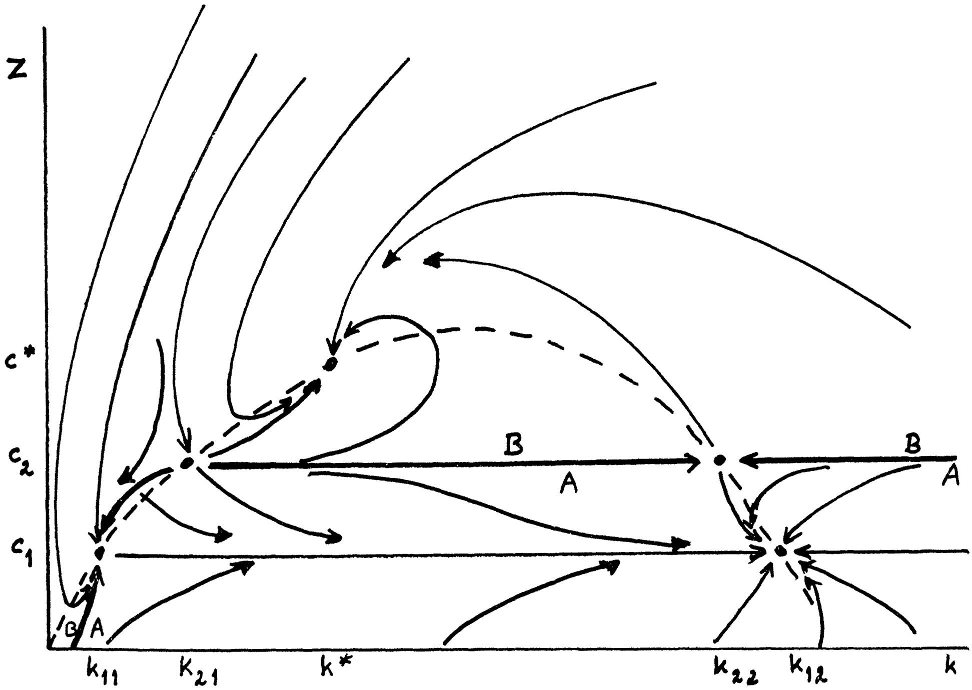

FIGURE 19

$c_1 < c_2 < c^*$.

and 0·1, and λ between 0·01 and 0·02. Such variations caused no departure from the pattern of roots that occurred when $\delta = \lambda = 0{\cdot}01$). The parameter ρ, which controls the length of the system's " memory ", was varied between 0·1 and 0·3, and sometimes beyond this range. The values 0·1 and 0·3 were felt to span the range of likely ρ-values. When $\rho = 0{\cdot}1$, $\exp(-30\rho) = 0{\cdot}0498$ and $\exp(-10\rho) = 0{\cdot}368$: consumption that occurred a generation back is given five per cent of the weight of current consumption, and consumption that occurred ten years back, one third the weight. When $\rho = 0{\cdot}3$, $\exp(-30\rho) = 0{\cdot}000123$ and $\exp(-10\rho) = 0{\cdot}0498$: the consumption of a generation back is given virtually no weight, and that of ten years back, only five per cent of the weight of current consumption. For all solutions calculated, the modified golden rule solution has $k^* = 68{\cdot}0$, $c^* = 3{\cdot}40$.

8.1 *The Function* $-\exp-(c-z)-c^{-\frac{1}{2}}$

With this function $|u_z|/|u_c| < 1$ for all c and z. Setting $c = z$ and increasing both together always increases welfare, as in Figure 3. Hence, there are no stationary points with satiation. It is the conventional golden rule capital-labour ratio that gives the highest indefinitely-maintainable level of instantaneous welfare. For this function, we have adjacent complementarity, $\gamma < 0$. The roots to (45) at a modified golden rule solution are as follows for various values of ρ, $(i^2 = -1)$

$$\rho = 0{\cdot}05{:}\ +0{\cdot}023 \pm 0{\cdot}014i{:}\ -0{\cdot}013 \pm 0{\cdot}014i$$

$$\rho = 0{\cdot}10{:}\ +0{\cdot}028 \pm 0{\cdot}017i{:}\ -0{\cdot}018 \pm 0{\cdot}017i$$

$$\rho = 0{\cdot}20{:}\ +0{\cdot}038 \pm 0{\cdot}018i{:}\ -0{\cdot}028 \pm 0{\cdot}018i$$

$$\rho = 0{\cdot}30{:}\ +0{\cdot}048 \pm 0{\cdot}014i{:}\ -0{\cdot}038 \pm 0{\cdot}014i$$

In all cases, two roots have negative real parts, so that there is a plane in the four-dimensional phase space from within which the stationary solution can be approached. Hence, it is locally stable in the sense that if the initial values $k(0)$ and $z(0)$ lie within a certain neighbourhood of it, then there exist prices $q(0)$ and $p(0)$ such that the differential equations starting from the point $(k(0), z(0), q(0), p(0))$ converge asymptotically to the solution. It is shown below that if $k(0)$ and $z(0)$ lie within this neighbourhood, an optimal path will follow one of these convergent trajectories. All components of the motion in the neighbourhood of (k^*, z^*) are oscillatory, giving rise to both damped and anti-damped cyclical movements. Correct choice of the initial prices can ensure that only the damped movements are effective: the optimal path will oscillate, converging to (k^*, z^*), the modified golden rule stationary solution, as in Figure 11. That the paths shown in Figure 11 are indeed optimal paths, at least for initial points " near " (k^*, z^*), follows from Theorem 2. It is clear that such paths satisfy the conditions of Theorem 2: in particular, since $q>0$ at (k^*, z^*), it must (from (27-2)) be positive on any path leading to this point.

One can calculate the periods of the cycles implied by the complex roots: these periods (in years) are shown for various ρ values in the following table. The column headed " damping " gives the ratio of the amplitude at the end of a cycle to that at the beginning of the cycle.

ρ	Period	Damping
0·05	438	$0{\cdot}32\times 10^{-2}$
0·10	361	$0{\cdot}13\times 10^{-2}$
0·20	342	$0{\cdot}65\times 10^{-4}$
0·30	455	$0{\cdot}38\times 10^{-8}$

The cycles are extremely lengthy and very heavily damped: the error involved in approximating them by a smoothly rising or falling path might not be too great.

It is interesting to discuss in intuitive terms why oscillatory behaviour arises, and why it should become more heavily damped as ρ rises and the " memory " shortens. The explanation seems to lie in the fact that utility is derived from the difference between c and z, and that if c is increased quickly, z follows only with a lag. Hence, the following cycle may be advantageous in raising the term $\int -\exp-(c-z)$: at first increase c quickly, z lags behind so that $(c-z)$ increases. Now maintain c constant as z catches up, and then reduce c slowly. By making the reduction gradual, the amount by which z overshoots c can be minimized: after the reduction, the cycle begins again. On the upward part of this cycle, $(c-z)$ can be made large and positive: on the downward part, it can be prevented from becoming large and negative. Under certain circumstances this behaviour gives a larger integral of satisfaction than is obtained along a path where c and z move steadily. As ρ rises, the responsiveness of z to current consumption rises, making it more difficult to increase the gap between c and z: the attractions of oscillatory behaviour are thus reduced. Thus accords with common sense, as one would expect optimal paths in the present model to tend to the " usual " Ramsey paths as ρ gets very large.

8.2 *The Function* $(c/z)^{\frac{1}{2}}$

With this function, $|u_z|/|u_c| = 1$ when $c = z$: its contours are the borderline between the cases shown in Figures 3 and 4 and are rays proceeding from the origin. The function exhibits adjacent complementarity, but is not concave. Moving along the $c = z$ ray in the $z=c$ plane neither raises nor lowers instantaneous welfare: raising consumption and expected consumption in proportion leaves satisfaction unchanged. There are again no satiated stationary solutions and $q>0$ at the modified golden rule stationary solution. Earlier sections suggest that optimal paths will proceed towards this point, and the numerical results seem to confirm this. The roots of (48) at (k^*, z^*) are as follows:

$$\rho = 0{\cdot}10\text{: } +0{\cdot}042 \pm 0{\cdot}016i\text{: } -0{\cdot}032 \pm 0{\cdot}016i$$

$$\rho = 0{\cdot}20\text{: } +0{\cdot}052 \pm 0{\cdot}011i\text{: } -0{\cdot}042 \pm 0{\cdot}011i$$

$$\rho = 0{\cdot}30\text{: } +0{\cdot}070,\ +0{\cdot}049\text{: } -0{\cdot}039,\ -0{\cdot}060.$$

The stationary solution is once again approached via damped cycles for low ρ values.

ρ	Period	Damping
0·10	396	$0{\cdot}27 \times 10^{-5}$
0·20	570	$0{\cdot}40 \times 10^{-10}$

For $\rho = 0{\cdot}30$, the approach paths are as shown in Figure 10; these results seem to reinforce the argument that oscillatory behaviour is less likely for high ρ values.

8.3 *The Function* $(-z)^3 - c^{-\frac{1}{2}}$

In this case the contours are as shown in Figure 4, and raising c and z together along the $c-z$ ray eventually lowers social welfare. For this function, we have distant complementarity, $\gamma > 0$. There are now two satiated stationary points: some idea of the relative positions is conveyed by the following figures. $c^* = 3{\cdot}402$, $k^* = 68{\cdot}04$, and for all ρ values used, $c_1 \cong 0{\cdot}61$, $k_{11} \cong 0{\cdot}23$ and $k_{12} \cong 910$. Applying (67) we find that at each satiated solution there are two positive and two negative roots. Since $c_1 < c^*$, we would not in this case expect an optimal path to tend towards the modified golden rule solution, as the golden rule capital-labour ratio no longer gives the highest indefinitely maintainable level of satisfaction: indeed it gives a local minimum of this quantity. We have the motion shown in Figure 16.

Computations reveal the following roots at the satiated optimal stationary points. For $k_{11} \cong 0{\cdot}23$:

$$\rho = 0{\cdot}10;\ \mu = 0{\cdot}862,\ -0{\cdot}852,\ 0{\cdot}164,\ -0{\cdot}154;\ k_{11} = 0{\cdot}236$$

$$\rho = 0{\cdot}20;\ \mu = 0{\cdot}886,\ -0{\cdot}876,\ 0{\cdot}318,\ -0{\cdot}308;\ k_{11} = 0{\cdot}227$$

$$\rho = 0{\cdot}30;\ \mu = 0{\cdot}894,\ -0{\cdot}884,\ 0{\cdot}471,\ -0{\cdot}461;\ k_{11} = 0{\cdot}224$$

For $k_{12} \cong 910$:

$$\rho = 0{\cdot}10;\ \mu = -0{\cdot}006440,\ 0{\cdot}016440,\ 0{\cdot}164,\ -0{\cdot}154;\ k_{12} = 906$$

$$\rho = 0{\cdot}20;\ \mu = -0{\cdot}006443,\ 0{\cdot}016443,\ 0{\cdot}318,\ -0{\cdot}308;\ k_{12} = 907$$

$$\rho = 0{\cdot}30;\ \mu = -0{\cdot}006445,\ 0{\cdot}016445,\ 0{\cdot}471,\ -0{\cdot}461;\ k_{12} = 908$$

Since the results are similar for all cases, we calculated characteristic vectors only for $\rho = 0{\cdot}10$. The right-hand stationary solution $k_{12} = 906$ is in the interior of region A. The relevant roots are $\mu_1 = 0{\cdot}006440$ with characteristic vector $k' = 1$, $z' = q' = p' = 0$ and $\mu_2 = -0{\cdot}154$ with characteristic vector

$$k^4 = 1{\cdot}000,\ z^4 = -0{\cdot}269,\ q^4 = 0,\ p^4 = 3{\cdot}75.$$

At the left-hand stationary solution $k_{11} = 0{\cdot}236$, there are three relevant roots. The unstable root $\mu_1 = 0{\cdot}862$ carries motion into the interior of region A along the characteristic vector $k' = 1$, $z' = q' = p' = 0$. The stable root $\mu_4 = -0{\cdot}154$ has the characteristic vector $k^4 = 1{\cdot}000$, $z^4 = -1{\cdot}84$, $q^4 = 0$, $p^4 = 25{\cdot}6$, which forms the boundary between regions A and B. The stable root $\mu_2 = -0{\cdot}852$ approaches the stationary point from the interior of region B along the characteristic vector $k^2 = -1{\cdot}000$, $z^2 = 0{\cdot}228$, $q^2 = 4{\cdot}23$,

$p^2 = -0{\cdot}875$. Thus we have $q>0$ in the interior of region B. It would not be optimal to approach the stationary point along this vector in region A since we would have $q<0$ violating (29). Similarly it is not optimal to move into region B along the vector $k' = 1$, $z' = q' = p' = 0$, since we must eventually enter Phase I, and would have $q<0$ in the interior of Phase I. As indicated above, k_{11} and k_{12} differ substantially from k^* in the case investigated, so that there is here a marked difference from the results in papers such as [2] and [13]: only Kurz [11] has found a pattern with any resemblance to the present one.

9. CONCLUSIONS

The results presented above confirm that altering the objective function in the manner suggested does indeed alter the nature of an optimal path, and the alteration may be quite substantial. If satiation is excluded, there is still a unique optimal steady state—the usual modified golden rule. The differences are in the optimal behaviour around that stationary point. Under independent utility, the optimal current consumption depends only on the current capital-labour ratio, k, and the modified golden rule is approached monotonically in k. Under dependent utility, the optimal current consumption is also affected by past levels of consumption. Now k may overshoot its equilibrium value before settling down. Indeed, it may not even settle down, but rather oscillate endlessly in a limit cycle.

If an increase in a uniformly maintained consumption level eventually decreases utility, there may be satiated optimal stationary points in addition to or in place of the modified golden rule. Under satiation it is possible that an increase in the amount of capital available to the economy will make it no better off. This is reflected in the fact that the price of capital becomes zero in such a case. It is not surprising that such a possibility might fundamentally alter the nature of an optimal programme of capital accumulation.

Of course, the model discussed is, like any simple model, deficient in a number of respects. One deficiency is that there is no provision for disposing of output or capital. If, for example, the equality in equation (4) were replaced by a weak inequality, then there might well be situations in which it becomes optimal for a country to give away some part of its output. Another qualification derives from the fact that expected consumption levels may not be determined only by past consumption, but may have an exogenous upward trend arising in affluent societies from the activities of advertisers, and in developing countries, from international comparisons.

APPENDIX A

Theorem 1. *Under assumptions* (*P.* 4), (*T.* 1) *and* (*T.* 2) *there is a unique consumption path that maximizes* (3) *subject to* (4), (5), (6), (7).

Proof. Existence: We consider the nonlinear process in R^3

$$\dot{w} = e^{-\delta t}u(c, z) \qquad \ldots(A\text{-}1)$$

$$\dot{z} = \rho(c-z) \qquad \ldots(6)$$

$$\dot{k} = f(k)-\lambda k-c \qquad \ldots(4)$$

with initial state $w(0) = 0$, $z(0) = z_0>0$, $k(0) = k_0>0$. The admissible controls are all measurable functions $c(t)$ with

$$0 \leqq c \leqq f(k). \qquad \ldots(5)$$

We show next that the system is bounded. Note first that there exists $\hat{k}>0$ such that $f(\hat{k}) = \lambda\hat{k}$, and $f(k)-\lambda k<0$ for $k>\hat{k}$. Now let

$$\bar{k} = \max\,[\hat{k}, k_0],\ \underline{k}_T = k_0e^{-\lambda T}>0. \qquad \ldots(A\text{-}2)$$

Since $-\lambda k \leqq \dot{k} \leqq f(k)-\lambda k$, we have

$$\underline{k}_T \leqq k(t) \leqq \bar{k} \text{ for all } 0 \leqq t \leqq T.$$

Then $0 \leqq c(t) \leqq f(\bar{k})$ for all $0 \leqq t \leqq T$. Let

$$\bar{z} = \max [f(\bar{k}), z_0], \ \underline{z}_T = z_0 e^{-\rho T} > 0. \qquad \ldots(A\text{-}3)$$

Since $-\rho z \leqq \dot{z} \leqq \rho f(\bar{k}) - \rho z$, we have

$$\underline{z}_T \leqq z(t) \leqq \bar{z} \text{ for all } 0 \leqq t \leqq T.$$

Let

$$\bar{u} = \max_{\substack{0 \leqq c \leqq \bar{z} \\ 0 \leqq z \leqq \bar{z}}} u(c, z), \ \bar{w} = \bar{u}/\delta. \qquad \ldots(A\text{-}4)$$

Then $w(t) \leqq \bar{w}$ for all $0 \leqq t \leqq T$. Let

$$V(w, z, k, t) \equiv \{(\dot{w}, \dot{z}, \dot{k}) | \ \dot{w} \leqq e^{-\delta t} u(c, z), \ \dot{z} = \rho(c-z), \ \dot{k} = f(k) - \lambda k - c, \ 0 \leqq c \leqq f(k)\}. \qquad \ldots(A\text{-}5)$$

The set $V(w, z, k, t)$ of velocity vectors is convex, closed, bounded in $\dot{z}$ and $\dot{k}$, and bounded above in $\dot{w}$. Let

$$S(t) \equiv \{(w, z, k) | \ w \leqq \int_0^t e^{-\delta\tau} u(c, z) d\tau, \ z = z_0 + \int_0^t \rho(c-z) d\tau,$$

$$k = k_0 + \int_0^t [f(k) - \lambda k - c] d\tau \text{ for some admissible control } c(\tau), \ 0 \leqq \tau \leqq t\}. \qquad \ldots(A\text{-}6)$$

Then by a theorem of Lee and Markus [12; p. 242, Theorem 2], the attainable set $S(t)$ is closed, bounded in z and k, and bounded above in w, and varies continuously on $0 \leqq t \leqq T$.

We know that $J[c(\cdot)] = \lim_{t \to \infty} w(t) \leqq \bar{w}$ for all admissible controls. Let $M = \sup J[c(\cdot)]$ over all admissible controls. Consider a sequence of admissible paths

$$\{w^i(t), z^i(t), k^i(t), c^i(t); \ t \geqq 0\} \ i = 1, 2, \ldots$$

such that

$$\lim_{i \to \infty} J[c^i(\cdot)] = M.$$

For any $\varepsilon > 0$, there exists $I > 0$ such that $i > I$ implies

$$J[c^i(\cdot)] > M - \frac{\varepsilon}{2}.$$

Let $T = (\ln 2\bar{u}/\delta\varepsilon)\delta$. Then $i > I$ implies

$$M < J[c^i(\cdot)] + \frac{\varepsilon}{2} = w^i(T) + \int_T^\infty e^{-\delta t} u(c^i, z^i) dt + \frac{\varepsilon}{2}$$

$$\leqq w^i(T) + e^{-\delta T} \frac{\bar{u}}{\delta} + \frac{\varepsilon}{2} = w^i(T) + \varepsilon.$$

For each T, we can choose a subsequence so that

$$w^i(T) \to w^0(T), \ z^i(T) \to z^0(T), \ k^i(T) \to k^0(T).$$

Then $[w^0(T), z^0(T), k^0(T)] \in S(T)$ for all $T \geqq 0$. Therefore, $\{w^0(t), z^0(t), k^0(t), c^0(t); t \geqq 0\}$ is an admissible path. But $w^0(T) > M - \varepsilon$. Therefore,

$$J[c^0(\cdot)] = \lim_{T \to \infty} w^0(T) = M. \qquad \ldots(A\text{-}7)$$

Thus it is also an optimal path.

This completes the proof of existence. Uniqueness can be established in a straightforward way using the concavity assumptions (P. 4) and (T. 1).

APPENDIX B

We wish to determine the derivatives of the optimal decision rule $c(z, k)$ described in Sections 6 and 7. If we linearize the two simultaneous differential equations (4) and (6) under the decision rule, we obtain in the neighbourhood of a stationary point (z^s, k^s)

$$\begin{bmatrix} \dot{z} \\ \dot{k} \end{bmatrix} = \begin{bmatrix} \rho c_z - \rho & \rho c_k \\ -c_z & \delta - c_k \end{bmatrix} \begin{bmatrix} z - z^s \\ k - k^s \end{bmatrix}. \qquad \ldots(B\text{-}1)$$

This linear system has the characteristic polynomial

$$\mu^2 + [c_k + \rho(1 - c_z) - \delta]\mu + \rho[c_k - \delta(1 - c_z)] = 0. \qquad \ldots(B\text{-}2)$$

Knowing the appropriate characteristic roots μ_i and μ_j, we can solve for the derivatives of the optimal decision rule

$$c_k = \frac{(f' - \lambda - \mu_i)(f' - \lambda - \mu_j)}{(f' - \lambda + \rho)} \qquad \ldots(B\text{-}3)$$

$$c_z = \frac{(\rho + \mu_i)(\rho + \mu_j)}{\rho(f' - \lambda + \rho)}. \qquad \ldots(B\text{-}4)$$

The two appropriate roots are generally the ones with the smallest real parts, unless one of these can be shown to be ineligible.

If the modified golden rule (z^*, k^*) has $q > 0$, it is an optimal stationary solution. The appropriate roots are

$$\mu_{i,j} = \frac{\delta}{2} - \tfrac{1}{2}\sqrt{\frac{\delta^2}{2} + \sigma + \sqrt{\frac{\delta^4}{4} + \delta^2\sigma + 4\tau}} \pm \tfrac{1}{2}\sqrt{\frac{\delta^2}{2} + \sigma - \sqrt{\frac{\delta^4}{4} + \delta^2\sigma + 4\tau}} \qquad \ldots(B\text{-}5)$$

Since $f' - \lambda = \delta$, we have $f' - \lambda - \mu_i > \delta/2 > 0$, so clearly $c_k(z^*, k^*) > 0$. Substituting (B-5) into (B-4) we obtain

$$c_z = \frac{1}{\rho(\rho + \delta)}\left[\left(\frac{\delta}{2} + \rho\right)^2 - \left(\frac{\delta}{2} + \rho\right)\sqrt{\frac{\delta^2}{2} + \sigma + \sqrt{\frac{\delta^4}{4} + \delta^2\sigma + 4\tau}} + \tfrac{1}{2}\sqrt{\frac{\delta^4}{4} + \delta^2\sigma + 4\tau}\right]. \qquad \ldots(B\text{-}6)$$

Recalling the definitions (49) and (50),

$$\left(\frac{\delta}{2} + \rho\right)^2 \left(\frac{\delta^2}{4} + \beta + \rho\gamma\right) \gtreqless \tfrac{1}{4}\left(\frac{\delta^4}{4} + \delta^2\sigma + 4\tau\right) \text{ as } \gamma \gtreqless 0.$$

Then

$$\left(\frac{\delta}{2} + \rho\right)^2 \left[\frac{\delta^2}{2} + \delta\rho + \rho^2 + \beta + \rho\gamma + \sqrt{\frac{\delta^4}{4} + \delta^2\sigma + 4\tau}\right] \gtreqless \left[\left(\frac{\delta}{2} + \rho\right)^2 + \tfrac{1}{2}\sqrt{\frac{\delta^4}{4} + \delta^2\sigma + 4\tau}\right]^2.$$

Taking positive square roots, we find

$$\left(\frac{\delta}{2} + \rho\right)^2 + \tfrac{1}{2}\sqrt{\frac{\delta^4}{4} + \delta^2\sigma + 4\tau} \gtreqless \left(\frac{\delta}{2} + \rho\right)\sqrt{\frac{\delta^2}{2} + \sigma + \sqrt{\frac{\delta^4}{4} + \delta^2\sigma + 4\tau}}. \qquad \ldots(B\text{-}7)$$

Substituting (B-7) into (B-6) we find that

$$c_z \gtreqless 0 \text{ as } \gamma \gtreqless 0.$$

By assumption (P. 4), we have

$$\sigma - \beta \geq -\frac{\delta^2}{4}.$$

Then

$$(\rho\delta+\rho^2)\left[\frac{\delta^2}{2}+\sigma-\beta+\sqrt{\frac{\delta^4}{4}+\delta^2\sigma+4\tau}\right]>\left[\frac{\delta^2}{4}+\tfrac{1}{2}\sqrt{\frac{\delta^4}{4}+\delta^2\sigma+4\tau}\right]^2.$$

Taking positive square roots, we find

$$\left(\frac{\delta}{2}+\rho\right)\sqrt{\frac{\delta^2}{2}+\sigma+\sqrt{\frac{\delta^4}{4}+\delta^2\sigma+4\tau}}>\left(\frac{\delta^2}{4}+\rho\delta+\rho^2+\tfrac{1}{2}\sqrt{\frac{\delta^4}{4}+\delta^2\sigma+4\tau}\right)-(\rho\delta+\rho^2). \quad \text{...}(B\text{-}8)$$

Comparing (B-8) with (B-6) we see that

$$c_z<1.$$

If there are satiated optimal stationary solutions (c_i, k_{ij}), we have seen in Section 7 that the characteristic roots are

$$\begin{aligned}\mu_1&=f'-\lambda,\\ \mu_2&=\delta+\lambda-f',\\ \mu_3&=\frac{\delta}{2}+\tfrac{1}{2}\sqrt{\delta^2+4\rho(\rho+\delta+\gamma)},\\ \mu_4&=\frac{\delta}{2}-\tfrac{1}{2}\sqrt{\delta^2+4\rho(\rho+\delta+\gamma)}.\end{aligned} \quad \text{...}(67)$$

One of the two smallest roots is always μ_4, the other is μ_2 if $f'-\lambda>\delta/2$ and μ_1 if $f'-\lambda<\delta/2$. But even if μ_2 is one of the smallest roots, it is ineligible in the region to the right of the characteristic vector (z^4, k^4), since negative values of q are not optimal (equation (29)). Thus at stationary points where $f'-\lambda>\delta/2$, the appropriate roots are (μ_2, μ_4) in the half-neighbourhood to the left of (z^4, k^4) and (μ_1, μ_4) in the half-neighbourhood to the right of (z^4, k^4). At stationary points where $f'-\lambda<\delta/2$, the appropriate roots are (μ_1, μ_4).

If the appropriate roots are (μ_1, μ_4), equations (B-3) and (B-4) become

$$c_k^A(c_i, k_{ij})=0, \quad \text{...}(70)$$

$$c_z^A(c_i, k_{ij})=1+\frac{\mu_4}{\rho}=\frac{1}{\rho}\left[\left(\frac{\delta}{2}+\rho\right)-\sqrt{\left(\frac{\delta}{2}+\rho\right)^2+\rho\gamma}\right]. \quad \text{...}(B\text{-}9)$$

Clearly

$$c_z^A(c_i, k_{ij})\lesseqgtr 1 \text{ as } \mu_4\lesseqgtr 0 \text{ as } q'(c_i)\gtreqless 0, \quad \text{...}(72)$$

and

$$c_z^A(c_i, k_{ij})\gtreqless 0 \text{ as } \gamma\lesseqgtr 0. \quad \text{...}(73)$$

If the appropriate roots are (μ_2, μ_4), equations (B-3) and (B-4) become

$$c_k^B(c_i, k_{i1})=\frac{2\left(f'-\lambda-\dfrac{\delta}{2}\right)\left(f'-\lambda-\dfrac{\delta}{2}+\tfrac{1}{2}\sqrt{\delta^2+4\rho(\rho+\delta+\gamma)}\right)}{f'-\lambda+\rho} \quad \text{...}(B\text{-}10)$$

$$c_z^B(c_i, k_{i1})=\frac{1}{\rho}\left[\left(\frac{\delta}{2}+\rho\right)-\sqrt{\left(\frac{\delta}{2}+\rho\right)^2+\rho\gamma}\right]\frac{(\delta+\lambda-f'+\rho)}{(f'-\lambda+\rho)}. \quad \text{...}(B\text{-}11)$$

Since these derivatives apply only for $f'-\lambda > \delta/2$, we have

$$c_k^B(c_i, k_{i1}) > 0, \quad \text{...(71)}$$

$$\frac{c_z^B(c_i, k_{i1})}{c_z^A(c_i, k_{i1})} = \frac{\left(\frac{\delta}{2}+\rho\right) - \left(f'-\lambda-\frac{\delta}{2}\right)}{\left(\frac{\delta}{2}+\rho\right) + \left(f'-\lambda-\frac{\delta}{2}\right)} < 1, \quad \text{...(74)}$$

$$\frac{c_z^B(c_i, k_{i1})}{c_z^A(c_i, k_{i1})} \gtreqless 0 \text{ as } f'-\lambda \lesseqgtr \delta+\rho. \quad \text{...(75)}$$

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